

AGING IN REVERSIBLE DYNAMICS OF DISORDERED SYSTEMS.
II. emergence of the arcsine law in the random hopping
time dynamics of the REM.

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Abstract: Applying the new tools developed in [G1], we investigate the arcsine aging regime of the random hopping time dynamics of the REM. Our results are optimal in several ways. They cover the full time-scale and temperature domain where this phenomenon occurs. On this domain the limiting clock process and associated time correlation function are explicitly constructed. Finally, all convergence statements w.r.t. the law of the random environment are obtained in the strongest sense possible, except perhaps on the very last scales before equilibrium.

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1. Introduction.

1.1. The setting.

New tools for the study of the aging behavior of disordered systems were developed in [G1] and successfully applied to the study of the arcsine aging regime of Bouchaud's asymmetric trap model on the complete graph [B, BD, BRM]. We refer the reader to the introduction of [G1] for an overall presentation of the aging phenomenon. In this follow up paper we continue the investigation of the aging behavior of disordered systems, focusing on the the so-called random hopping time dynamics of Derrida's REM [D1,D2]. Our objectives here are twofold:

- 1) Establish the optimal time scale and temperature domain where the model exhibits an arcsine aging regime, striving for statements that are valid in the strongest sense possible w.r.t. the law of the random environment. Indeed, if the random hopping time dynamics (RHT) of the REM has been intensively studied, each existing contribution [BBG1, BBG2, BBG3, BC, CG] concerns specific choices of the time scale and temperature parameters, and a full, unifying picture was still missing.
- 2) Explain why and how the RHT dynamics of the REM and Bouchaud's REM-like trap model exhibit the same arcsine aging regime. More precisely, identify and isolate the mechanisms that, within the new technical framework of [G1], allow to reduce the study of (the clock process of) the REM to that of its trap version. This technical issue is a crucial step towards the understanding of the more interesting metropolis dynamics of the REM, [G2].

We now specify the model and succinctly recall the basics of arcsine aging – a detailed exposition can be found in [G1]. Denote by $\mathcal{V}_n = \{-1, 1\}^n$ the n -dimensional discrete cube, and by \mathcal{E}_n its edges set. On \mathcal{V}_n we construct a random landscape of traps (the random environment) by assigning to each site, x , the Boltzman weight of the REM, $\tau_n(x)$. Namely, given a parameter $\beta > 0$, called the inverse temperature, and a collection $(H_n(x), x \in \mathcal{V}_n)$ of independent standard gaussian random variables, we set $\tau_n(x) = \exp(-\beta\sqrt{n}H_n(x))$. (The dependence of $\tau_n(x)$ on β will be kept implicit.) The sequence $(\tau_n(x), x \in \mathcal{V}_n)$, $n \geq 1$, is defined on a common probability space denoted $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$.

The RHT dynamics in the landscape $(\tau_n(x), x \in \mathcal{V}_n)$ is a continuous time Markov chain $(X_n(t), t > 0)$ on \mathcal{V}_n that can be constructed as follows: let $(J_n(k), k \in \mathbb{N})$ be the simple random walk on \mathcal{V}_n with initial distribution μ_n and transition probabilities

$$p_n(x, y) = \frac{1}{n}, \quad \forall (x, y) \in \mathcal{E}_n, \quad (1.1)$$

and let the *clock process* be the partial sum process

$$\tilde{S}_n(k) = \sum_{i=0}^k \tau_n(J_n(i))e_{n,i}, \quad k \in \mathbb{N}, \quad (1.2)$$

where $(e_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$ is a family of independent mean one exponential random variables, independent of J_n ; then

$$X_n(t) = J_n(i) \quad \text{if} \quad \tilde{S}_n(i) \leq t < \tilde{S}_n(i+1) \quad \text{for some } i. \quad (1.3)$$

This defines X_n in terms of its jump chain, J_n , and holding times, $\tau_n(x)$ being the mean value of the exponential holding time at x . Equivalently, X_n is the chain with initial distribution μ_n and jump rates $\lambda_n(x, y) = (n\tau_n(x))^{-1}$, $(x, y) \in \mathcal{E}_n$, $x \neq y$. This last description makes it easy to check that X_n is a Glauber dynamics, namely, that it is reversible with respect to the measure (the Gibbs measure of the REM) defined on \mathcal{V}_n by

$$\mathcal{G}_n(x) = \frac{\tau_n(x)}{\sum_{x \in \mathcal{V}_n} \tau_n(x)}, \quad x \in \mathcal{V}_n. \quad (1.4)$$

The model we referred to as the REM-like trap model was proposed by Bouchaud as an approximation of the aging dynamics of the REM (see [BBG1] for details on this derivation). It is a Markov chain X'_n on $\mathcal{V}'_n = \{1, \dots, n\}$ with jump rates $\lambda'_n(x, y) = (n\tau'(x))^{-1}$, $(x, y) \in \mathcal{V}'_n \times \mathcal{V}'_n$, $x \neq y$, where $(\tau'(x), x \in \mathcal{V}'_n)$ are i.i.d. r.v. in the domain of attraction of a positive stable law with index $0 < \alpha < 1^2$.

To study the aging behavior of X_n we need to choose three ingredients: 1) an initial distribution, μ_n ; 2) a time scale of observation, c_n ; and 3) a time-time correlation function, $\mathcal{C}_n(t, s)$, $t, s \geq 0$: this is a function that quantifies the correlation between the state of the system at time t , $X_n(c_n t)$, and its state at time $t + s$, $X_n(c_n(t + s))$. We will say that the process X_n has an arcsine aging regime of parameter $0 < \alpha < 1$ whenever one can find a time-time correlation function such that, denoting by $\text{Asl}_\alpha(\cdot)$ the generalized arcsine distribution function of parameter α ,

$$\text{Asl}_\alpha(u) = \frac{\sin \alpha \pi}{\pi} \int_0^u (1-x)^{-\alpha} x^{-1+\alpha} dx, \quad 0 < \alpha < 1, \quad (1.5)$$

one of the following relations³ holds true,

$$\lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_\alpha(1/1 + \rho), \quad (1.6)$$

$$\lim_{n \rightarrow \infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_\alpha(1/1 + \rho), \quad t > 0 \text{ arbitrary}, \quad (1.7)$$

for all $\rho \geq 0$, and some convergence mode w.r.t. the probability law \mathbb{P} of the random landscape. It is today well understood that the existence of an arcsine aging regime is governed by Dynkin and Lamperti's arcsine law for subordinators, applied to the limiting, appropriately re-scaled, clock process: arcsine aging will be present when the re-scaled clock process converges to a subordinator whose Lévy measure satisfies the slow variation conditions of the Dynkin-Lamperti Theorem⁴. In this light a natural indicator of arcsine aging, and the one we will choose, is:

$$\mathcal{C}_n(t, s) = \mathcal{P}_{\mu_n} \left(\left\{ c_n^{-1} \tilde{S}_n(k), k \in \mathbb{N} \right\} \cap (t, t + s) = \emptyset \right), \quad 0 \leq t < t + s. \quad (1.8)$$

²The model we just described is sometimes called Bouchaud's trap model on the complete graph. It is obtained by setting $a = 0$ in Bouchaud's asymmetric trap model on the complete graph studied in [G1]. In Bouchaud's original model, the following choices are made: $\tau'(x) = e^{E_x/\alpha}$, where $(E_x, x \in \mathcal{V}'_n)$ are i.i.d. mean one exponential r.v.'s, and $\alpha = \sqrt{2 \log 2 / \beta}$, $\beta > 0$.

³These are the only two limiting procedure relevant for the REM. A third one was present in Bouchaud's REM-like trap model: $\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_\alpha(1/1 + \rho)$.

⁴See e.g. Appendix A.2.1 of [G1] for a statement of the latter.

where \mathcal{P}_{μ_n} denotes the law of X_n with initial distribution μ_n . In words, this is the probability that the range of the re-scaled clock process $c_n^{-1}\tilde{S}_n$ does not intersect the time interval $(t, t+s)$. The initial distribution is taken to be the invariant measure of the jump chain; that is we set $\mu_n = \pi_n$, where

$$\pi_n(x) = 2^{-n}, \quad x \in \mathcal{V}_n. \quad (1.9)$$

We will see in later that this choice is generic⁵.

We now proceed to state our results. Subsection 1.2 contains the results on the convergence of \mathcal{C}_n . Their parent results on the convergence of the re-scaled clock process will be stated next, in Subsection 1.3. The closing Subsection 1.4 is devoted to the method of proof: we recall the needed results from [G1] and elaborate on the strategy of the proofs.

1.2. Convergence of the time-time correlation \mathcal{C}_n .

To keep the notation of this paper consistent with those of [G1] we set $c_n = r_n$ and call r_n a space scale.⁶ We will distinguish three types of space scales: the *short* scales, the *intermediate* scales and the *extreme* scales. Let b_n be defined through

$$b_n \mathbb{P}(\tau_n(x) \geq r_n) = 1, \quad (1.10)$$

and set $m_n = \log b_n / \log 2$.

Definition 1.1: We say that a diverging sequence r_n is:

- (i) a *short space scale* if m_n is a diverging sequence such that

$$\frac{m_n}{n} = o(1), \quad (1.11)$$

- (ii) an *intermediate space scale* if there exists $0 < \varepsilon \leq 1$ such that

$$\frac{m_n}{n} \sim \varepsilon \quad \text{and} \quad \frac{2^{m_n}}{2^n} = o(1), \quad (1.12)$$

- (iii) an *extreme space scale* if there exists $0 < \varepsilon' \leq 1$ such that

$$\frac{2^{m_n}}{2^n} \sim \varepsilon'. \quad (1.13)$$

Note that Definition 1.1 allows to classify all sequences b_n such that $1 \ll b_n \leq 2^n$, leaving out constant sequences only. For any of the above space scale set $\varepsilon = \lim_{n \rightarrow \infty} \frac{m_n}{n}$. Thus $\varepsilon = 0$ if r_n is a short space scale, $0 < \varepsilon \leq 1$ if r_n is an intermediate space scale, and $\varepsilon = 1$ if r_n is an extreme space scale. For $0 \leq \varepsilon \leq 1$ and $0 < \beta < \infty$, define

$$\begin{aligned} \beta_c(\varepsilon) &= \sqrt{\varepsilon 2 \log 2}, \\ \alpha(\varepsilon) &= \beta_c(\varepsilon) / \beta, \end{aligned} \quad (1.14)$$

⁵See the remark below Proposition 4.1.

⁶We saw in [G1] (see the paragraph above Definition 4.1) that, generally speaking, measuring time on scale c_n for X_n corresponds to re-scaling the traps $\tau_n(x)$ by a certain amount r_n . It is clear from (1.3) that here space and time scales coincide.

and write $\beta_c \equiv \beta_c(1)$ and $\alpha \equiv \alpha(1)$.

Before stating our two theorems let us outline their content. Theorem 1.2 establishes that arcsine aging is not present on short scales, where $\varepsilon = 0$, and emerges on intermediate scales, as ε becomes positive. There, it is present in the entire time scale and temperature domain $\{0 < \varepsilon \leq 1; 0 < \alpha(\varepsilon) < 1\}$, and gets interrupted on the critical temperature line $\{0 < \varepsilon \leq 1; \alpha(\varepsilon) = 1\}$. These results hold true for all time $t > 0$, that is, (1.7) prevails. On extreme scales, where $\varepsilon = 1$, arcsine aging is still present on the entire low temperature line $\{\varepsilon = 1; 0 < \alpha \equiv \alpha(1) < 1\}$, but in the limit $t \rightarrow 0$ only, that is, along the double limiting procedure of (1.6). These are the very last (and longest) scales where arcsine aging occurs before interruption. Indeed, as t increases from 0 to ∞ , the system moves out of its arcsine aging regime and crosses over to its stationarity regime. This last statement is the content of Theorem 1.3. Clearly all these convergence results must hold in some sense w.r.t. the law \mathbb{P} of the random landscape. We now make this precise.

Theorem 1.2: Take $\mu_n = \pi_n$.

- (i) Let r_n be a short space scale and, given any $0 < \beta < \infty$, assume that $\frac{1}{\beta n} \log r_n \geq 4\sqrt{\frac{\log n}{n}}$. Then, \mathbb{P} -almost surely, for all $t \geq 0$ and all $\rho > 0$,

$$\lim_{n \rightarrow \infty} C_n(t, \rho t) = 1. \quad (1.15)$$

- (ii) Let r_n be an intermediate space scale. For all $0 < \varepsilon \leq 1$ and all $0 < \beta < \infty$ such that $0 < \alpha(\varepsilon) \leq 1$ the following holds, \mathbb{P} -almost surely if $\frac{2^m n}{2^n} \log n = o(1)$, and in \mathbb{P} -probability otherwise: for all $t \geq 0$ and all $\rho > 0$,

$$\lim_{n \rightarrow \infty} C_n(t, \rho t) = \begin{cases} \text{Asl}_{\alpha(\varepsilon)}(1/1 + \rho), & \text{if } \alpha(\varepsilon) < 1, \\ 0, & \text{if } \alpha(\varepsilon) = 1. \end{cases} \quad (1.16)$$

- (iii) Let r_n be an extreme space scale. For all $\beta_c < \beta < \infty$, for all $\rho > 0$, in \mathbb{P} -probability,

$$\lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} C_n(t, \rho t) = \text{Asl}_{\alpha}(1/1 + \rho). \quad (1.17)$$

The next theorem shows that on extreme space (equivalently time) scales, taking the limit $n \rightarrow \infty$ first, the process reaches stationarity as $t \rightarrow \infty$ in the sense that the limiting time-time correlation function is the same as that of the process X_n started in its invariant measure, \mathcal{G}_n . To state this we need a little notation. Let $\text{PRM}(\mu)$ be the Poisson random measure on $(0, \infty)$ with mean measure defined through

$$\mu(x, \infty) = x^{-\alpha}, \quad x > 0, \quad (1.18)$$

and with marks $\{\gamma_k\}$. Define

$$\mathcal{C}_{\infty}^{sta}(s) = \sum_{k=1}^{\infty} \frac{\gamma_k}{\sum_{k=1}^{\infty} \gamma_k} e^{-s/\gamma_k}, \quad s \geq 0. \quad (1.19)$$

Theorem 1.3: [Crossover to stationarity.] *Let r_n be an extreme space scale. The following holds for all $\beta > \beta_c$.*

(i) *If $\mu_n = \mathcal{G}_n$ then, for all $s \leq t < t + s$,*

$$\lim_{n \rightarrow \infty} \mathcal{C}_n(t, s) \stackrel{d}{=} \mathcal{C}_\infty^{sta}(s), \quad (1.20)$$

where $\stackrel{d}{=}$ denotes equality in distribution.

(ii) *If $\mu_n = \pi_n$, for all $s \geq 0$,*

$$\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{C}_n(t, s) \stackrel{d}{=} \mathcal{C}_\infty^{sta}(s). \quad (1.21)$$

Let us now put Theorem 1.2 in the context of earlier results. The very first aging results for the REM were obtained in [BBG1, BBG2] for a discrete time version of RHT dynamics considered here, on extreme scales. Rather than taking the double limiting procedure of (1.6) we considered a decreasing sequence of extreme time scales, $c_n(E)$, $E < 0$,⁷ and proved a statement of the form $\lim_{t \rightarrow \infty} \lim_{E \rightarrow -\infty} \lim_{n \rightarrow \infty} \mathcal{C}_{n,E}(t, \rho t) = \text{Asl}_\alpha(1/1 + \rho)$, in \mathbb{P} -probability (see Theorem 1 of [BBG2]). The general approach of the proof is that of renewal theory; technically, it relies on a refined knowledge of the metastable behavior of the process. The aging scheme based on the arcsine law for subordinators was proposed later in the landmark paper [BC] and applied to the study of intermediate scales, using potential theoretic tools. The existence of a \mathbb{P} -almost sure arcsine aging regime was proved in [BC] (see Theorem 3.1), on the subset $\{\frac{3}{4} \leq \varepsilon < 1; 0 < \alpha(\varepsilon) < 1\}$ of the time scale and temperature domain, and later extended to the domain $\{0 < \varepsilon < 1; 0 < \alpha(\varepsilon) < 1\}$ in [CG] (see Theorem 2.1). This still leaves out the case $\varepsilon = 1$ of the longest intermediate scales before extreme scales. Let us finally stress that arcsine aging is believed to be universally present in dynamics of mean-field spin glasses. This fact is strongly supported in [BBC] where the existence of an arcsine aging regime is for the first time proved in a model with correlations, namely the p -spin SK model, albeit only “in distribution”.

So far we said nothing about the nature of the convergence of the random time-time correlation function. Consider the last two assertions of Theorem 1.2. We see that statements that are valid \mathbb{P} -almost surely are turned into \mathbb{P} -probability statements across the line $\varepsilon = 1$ (more precisely, when the relation $\frac{2^{m_n}}{2^n} \log n = o(1)$ fails to be verified). A way to understand this transition is through the nature of the set $\mathcal{T}_n = \{x \in \mathcal{V}_n \mid \mathbb{P}(\tau_n(x) \geq r_n)\}$. When r_n is an intermediate space scale with $\varepsilon < 1$, \mathcal{T}_n simply is a huge, exponentially large site percolation cloud. In contrast, when r_n is an extreme space scale, \mathcal{T}_n is the very small set made of extreme $\tau_n(x)$'s, and this set has a particular probabilistic structure, namely, it resembles (for large enough n) a Poisson point process. The change in the nature of our convergence statements reflects this change in the nature of \mathcal{T}_n . This also explains why, technically, dealing with extreme scales (or very long intermediate scales) is intrinsically more arduous than dealing with the shorter intermediate scales, below the line $\varepsilon = 1$.

⁷Specifically, $c_n(E) \sim e^{\beta \sqrt{n} u_n(E)}$ where $u_n(E)$ is defined through $2^n \mathbb{P}(\tau_n(x) \geq c_n(E)) \sim \exp\{-e^{-E}\}$. Thus, by (1.10) and (1.13) of Definition 1.1, $r_n \equiv c_n(E)$ is an extreme space scale.

To conclude our discussion of Theorem 1.2 let us comment on the two boundary cases $\varepsilon = 0$ and $\alpha(\varepsilon) = 1$, $0 < \varepsilon \leq 1$, where the limiting time-time correlation function is trivial⁸. In both cases one might expect that some non linear normalization of the clock process in (1.8) will produce a non trivial limit. This was just shown to be true for short space scales where a new aging regime, coined *extremal aging*, had been identified [BGu, Gu]. The critical temperature case, where a phenomenon called “dynamical ultrametricity” is expected to take place [BB] will be treated elsewhere.

1.3. Convergence of the clock process.

We now state the results on the clock process from which Theorem 1.2 and Theorem 1.3 will be deduced. We will see that to each time scale c_n there correspond *auxiliary time scales* a_n such that the re-scaled process

$$S_n(t) = c_n^{-1} \tilde{S}_n(\lfloor a_n t \rfloor), \quad t \geq 0, \quad (1.22)$$

converges weakly to a subordinator. A main aspect of our method of proof is that the limiting subordinator is constructed explicitly. This will allow us to conclude, comparing to the results of [G1], that both on intermediate and extreme scales the limit is the same as in Bouchaud’s REM-like trap model⁹. This in turn implies that on those scales the two models have the same arcsine aging regimes.

Throughout the rest of this paper the arrow \Rightarrow denotes weak convergence in the space $D([0, \infty))$ of càdlàg functions on $[0, \infty)$ equipped with the Skorohod J_1 -topology¹⁰. We first settle the degenerate case of short scales. Let δ_∞ denote the Dirac point mass at infinity.

Proposition 1.4: [Short scales.] *Take $\mu_n = \pi_n$ and choose a_n s.t. $a_n \sim b_n$. Given any $0 < \beta < \infty$, if r_n is a short space scale that satisfies $\frac{1}{\beta n} \log r_n \geq 4\sqrt{\frac{\log n}{n}}$, then, \mathbb{P} -almost surely,*

$$S_n(\cdot) \Rightarrow S^{short}(\cdot), \quad (1.23)$$

where S^{short} is the degenerate subordinator of Lévy measure $\nu^{short} = \delta_\infty$.

Turning to intermediate scales, we have:

Proposition 1.5: [Intermediate scales.] *Take $\mu_n = \pi_n$ and choose a_n s.t. $a_n \sim b_n$. If r_n is an intermediate space scale then, for all $0 < \varepsilon \leq 1$ and all $0 < \beta < \infty$ such that $0 < \alpha(\varepsilon) \leq 1$, the following holds: \mathbb{P} -almost surely if $(2^{m_n} \log n)/2^n = o(1)$ and in \mathbb{P} -probability otherwise,*

$$S_n(\cdot) \Rightarrow S^{int}(\cdot), \quad (1.24)$$

where S^{int} is the subordinator whose Lévy measure, ν^{int} , is defined on $(0, \infty)$ through

$$\nu^{int}(u, \infty) = u^{-\alpha(\varepsilon)} \alpha(\varepsilon) \Gamma(\alpha(\varepsilon)), \quad u > 0. \quad (1.25)$$

⁸See also the paragraph after Proposition 1.5.

⁹Compare Proposition 1.5 and Proposition 1.6 below with, respectively, Proposition 4.8 and Proposition 4.9 of [G1].

¹⁰see e.g. [W] p. 83 for the definition of convergence in $D([0, \infty))$.

Note that both S^{short} and S^{int} are stable subordinators of index, respectively, $\alpha(0) = 0$ and $0 < \alpha(\varepsilon) \leq 1$. The cases $\alpha(0) = 0$ and $\alpha(\varepsilon) = 1$ are said to be degenerate in the sense that the range of the subordinator reduces to the single point 0 for the former, and is made of the entire positive line $[0, \infty)$ for the latter.

In the final case of extreme scales the limiting process no longer is a stable subordinator. Neither is it a deterministic process. As in [G1] this case is by far the more involved one¹¹. Recall that for μ defined in (1.18), $\{\gamma_k\}$ denote the marks of $\text{PRM}(\mu)$, and introduce the re-scaled landscape variables:

$$\gamma_n(x) = r_n^{-1} \tau(x), \quad x \in \mathcal{V}_n. \quad (1.26)$$

Proposition 1.6: [Extreme scales.] *Take $\mu_n = \pi_n$ and choose a_n s.t. $a_n \sim b_n$. If r_n is an extreme space scale then both the sequence of re-scaled landscapes $(\gamma_n(x), x \in \mathcal{V}_n)$, $n \geq 1$, and the marks of $\text{PRM}(\mu)$ can be represented on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that, in this representation, denoting by \mathbf{S}_n the corresponding re-scaled clock process (1.22), the following holds: for all $\beta_c < \beta < \infty$, \mathbf{P} -almost surely,*

$$\mathbf{S}_n(\cdot) \Rightarrow S^{ext}(\cdot), \quad (1.27)$$

where S^{ext} is the subordinator whose Lévy measure, ν^{ext} , is the random measure on $(0, \infty)$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ through

$$\nu^{ext}(u, \infty) = \varepsilon' \sum_{k=1}^{\infty} e^{-u/\gamma_k}, \quad u > 0, \quad (1.28)$$

ε' being defined in (1.13).

Although the limiting subordinator is not stable, the tail of the random Lévy measure ν^{ext} is regularly varying at 0^+ :

Lemma 1.7: *If $\beta > \beta_c$, then \mathbf{P} -almost surely, $\nu^{ext}(u, \infty) \sim \varepsilon' u^{-\alpha} \alpha \Gamma(\alpha)$ as $u \rightarrow 0^+$.*

1.4. Key tools and strategy.

The proofs of virtually all the results of the previous subsections rely on a key tool, Theorem 1.8, which we now state. This theorem simplifies results from [G1], but does not specialize them to the REM dynamics. It applies to any Markov chain X_n with graph structure $(\mathcal{V}_n, \mathcal{E}_n)$, jump chain J_n , and holding time parameters $(\lambda_n(x))_{x \in \mathcal{V}_n}$. This will allow us to compare the analysis of the REM dynamics to that of the REM-like trap model and, based on our understanding of the latter, to set up a strategy of proof. Let us also recall here that the results of [G1], and hence Theorem 1.8, are based on a powerful result by R. Durrett and S. Resnick [DuRe], that give conditions for partial sum processes of dependent random variables to converge to a subordinator.

¹¹See Proposition 4.9 and Section 7 of [G1]

Let us further denote by P_{μ_n} the law of J_n with initial distribution μ_n , and by $p_n(x, y)$ its one step transition probabilities. Thus, in the setting of Subsection 1.1, $p_n(x, y)$ is given by (1.1) and

$$\lambda_n(x) = (\tau_n(x))^{-1}, \quad \forall x \in \mathcal{V}_n. \quad (1.29)$$

Given sequences c_n and a_n , the clock process \tilde{S}_n and the re-scale clock process S_n are defined as

$$\begin{aligned} \tilde{S}_n(k) &= \sum_{i=0}^k (\lambda_n(J_n(i)))^{-1} e_{n,i}, \quad k \in \mathbb{N}, \\ S_n(t) &= c_n^{-1} \tilde{S}_n(\lfloor a_n t \rfloor), \quad t \geq 0. \end{aligned} \quad (1.30)$$

We now formulate four conditions for the sequence S_n to converge to a subordinator. We state these conditions for a fixed realization of the random landscape, i.e. for fixed $\omega \in \Omega^\tau$, and make this explicit by adding the superscript ω to all landscape dependent quantities.

Condition (A1). There exists a σ -finite measure ν on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge u) \nu(du) < \infty$, such that $\nu(u, \infty)$ is continuous, and such that, for all $t > 0$ and all $u > 0$,

$$P_{\mu_n}^\omega \left(\left| \sum_{j=1}^{\lfloor a_n t \rfloor} \sum_{x \in \mathcal{V}_n} p_n^\omega(J_n^\omega(j-1), x) e^{-uc_n \lambda_n^\omega(x)} - t \nu^\omega(u, \infty) \right| < \epsilon \right) = 1 - o(1), \quad \forall \epsilon > 0. \quad (1.31)$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P_{\mu_n}^\omega \left(\sum_{j=1}^{\lfloor a_n t \rfloor} \left[\sum_{x \in \mathcal{V}_n} p_n^\omega(J_n^\omega(j-1), x) e^{-uc_n \lambda_n^\omega(x)} \right]^2 < \epsilon \right) = 1 - o(1), \quad \forall \epsilon > 0. \quad (1.32)$$

Condition (A3). There exists a sequence of functions $\epsilon_n \geq 0$ satisfying $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \epsilon_n(\delta) = 0$ such that, for some $0 < \delta_0 \leq 1$, for all $0 < \delta \leq \delta_0$ and all $t > 0$,

$$E_{\mu_n}^\omega \left(\int_0^\delta du \sum_{j=1}^{\lfloor a_n t \rfloor} \sum_{x \in \mathcal{V}_n} p_n^\omega(J_n^\omega(j-1), x) e^{-uc_n \lambda_n^\omega(x)} \right) \leq t \epsilon_n(\delta). \quad (1.33)$$

Condition (A0').

$$\sum_{x \in \mathcal{V}_n} \mu_n^\omega(x) e^{-vc_n \lambda_n^\omega(x)} = o(1). \quad (1.34)$$

Theorem 1.8: For all sequences of initial distributions μ_n and all sequences a_n and c_n for which Conditions (A0'), (A1), (A2) and (A3) are verified, either \mathbb{P} -almost surely or in \mathbb{P} -probability, the following holds w.r.t. the same convergence mode: Let $\{(t_k, \xi_k)\}$ be the points of a Poisson random measure of intensity measure $dt \times d\nu$. We have,

$$S_n(\cdot) \Rightarrow S(\cdot) = \sum_{t_k \leq \cdot} \xi_k. \quad (1.35)$$

Moreover,

$$\lim_{n \rightarrow \infty} \mathcal{C}_n(t, s) = \mathcal{C}_\infty(t, s), \quad (1.36)$$

where

$$\mathcal{C}_\infty(t, s) = \mathcal{P}(\{S(u), u > 0\} \cap (t, t + s) = \emptyset), \quad 0 \leq t < t + s. \quad (1.37)$$

Proof: Theorem 1.8 results from the concatenation of Theorem 1.3 of [G1] and a trimmed version of Theorem 1.4 of [G1], where Condition (A0) is replaced by the more restrictive Condition (A0'). \square

Remark: Unlike in [G1] we did not separate the first steps of the clock process from the remaining ones. Instead, we made the simplifying assumption (see Condition (A0')) that the first step converges to zero. The question of more general initial distributions will be discussed elsewhere.

We see from Theorem 1.8 that the Lévy measure ν of the limiting subordinator in (1.35) is determined by Condition (A1). In order to prove that the limiting re-scaled clock process in the REM is the same as in the REM-like trap model, we first have to understand why and how Condition (A1) can be satisfied with the same measure ν . To this hand take $\mu_n = \pi_n$ for both models, π_n being the invariant measure of J_n , and set

$$\nu_n^{J,t}(u, \infty) = \sum_{j=1}^{\lfloor a_n t \rfloor} \sum_{x \in \mathcal{V}_n} p_n^\omega(J_n^\omega(j-1), x) e^{-uc_n \lambda_n^\omega(x)}, \quad u > 0. \quad (1.38)$$

Now note that, by reversibility, for both models,

$$E_{\pi_n} \nu_n^{J,t}(u, \infty) = \lfloor a_n t \rfloor \sum_{x \in \mathcal{V}_n} \pi_n(x) e^{-uc_n \lambda_n^\omega(x)}, \quad u > 0. \quad (1.39)$$

If we now specialize to the REM-like trap model the following two features will suffice to determine the measure ν :

- (i) $(\lambda_n^{-1}(x), x \in \mathcal{V}_n)$ are i.i.d. r.v.'s in the domain of attraction of a positive stable law with index $0 < \alpha < 1$, and
- (ii) $\nu_n^{J,t}(u, \infty) = E_\pi \nu_n^{J,t}(u, \infty)$.

Indeed, from (1.39) and (ii) one expects that, by independence and a concentration argument, $\nu(u, \infty) - \mathbb{E} E_\pi \nu_n^{J,t}(u, \infty) \downarrow 0$ as $n \uparrow \infty$ with \mathbb{P} -probability approaching one; using (i) then allows to compute $\mathbb{E} E_\pi \nu_n^{J,t}(u, \infty)$, and so $\nu = \lim_{n \rightarrow \infty} \mathbb{E} E_\pi \nu_n^{J,t}(u, \infty)$ provided that the limit exists¹².

The features (i) & (ii), that are put in by hand in REM-like trap model, are clearly not present in the RHT dynamics of the REM. However, if we can prove that (i') the re-scaled landscape $(c_n \lambda_n^{-1}(x), x \in \mathcal{V}_n)$ of the REM is heavy tailed (i.e., show that there exists sequences a_n and c_n such that $a_n \mathbb{P}(\lambda_n^{-1}(x) > uc_n) \sim u^{-\alpha}$, for some $0 < \alpha < 1$), and that,

¹²This reasoning applies on intermediate scales. A more refined argument is needed on extreme scales.

loosely speaking, (ii') the quantity $\nu_n^{J,t}(u, \infty)$ obeys an ergodic theorem with \mathbb{P} -probability close to one, then we should be able to reduce the analysis of the quantity $\nu_n^{J,t}(u, \infty)$ of the REM to that of the REM-like trap model. An analogous reasoning applies to Condition (A2). If (i') is known to hold from earlier works (at least on intermediate scales), how to justify (ii') is less immediate. To do this we will heavily draw on the specific properties of the jump chain J_n , in particular, on the fact that its trajectories do not depend on the randomness of the landscape. This is a key feature of the random hopping time dynamics dynamics, and the reason why its analysis is so much simpler than that of the usual metropolis dynamics [G2].

The rest of this paper is organized as follows. The heavy tailed nature of the trapping landscape is established in Section 2. Section 3 collects the properties of the jump chain J_n that will be needed in Section 4 to establish an 'ergodic theorem' for each of the sums appearing in Condition (A1) and Condition (A2) (called respectively $\nu_n^{J,t}(u, \infty)$ and $(\sigma_n^{J,t})^2(u, \infty)$). Once this is done it remains to establish the properties of the chain averaged sums: for this we separate the case of short and intermediate scales, treated in Section 5, from the case of extreme scales, dealt with in Section 6. The proofs of results obtained on short and intermediate scales (respec., extreme scales) are stated in Section 5, (respec., Section 6).

2. Properties of the landscape.

In this section we establish the needed properties of the re-scaled landscape variables $(r_n^{-1}\tau_n(x), x \in \mathcal{V}_n)$, and most importantly, the heavy tailed nature of their distribution. The notations of Subsection 1.2 (from (1.10) to (1.14)) are in force throughout this section. We assume that $0 < \beta < \infty$ is fixed, and as before, drop all dependence on β in the notation. For $u \geq 0$ set $G_n(u) = \mathbb{P}(\tau_n(x) > u)$. Since this is a continuous monotone decreasing function, it has a well defined inverse $G_n^{-1}(u) := \inf\{y \geq 0 : G_n(y) \leq u\}$.

For $v \geq 0$ set

$$h_n(v) = b_n G_n(r_n v). \quad (2.1)$$

Lemma 2.1: *Let r_n be any of the space scales of Definition 1.1.*

- (i) *For each fixed $\zeta > 0$ and all n sufficiently large so that $\zeta > r_n^{-1}$, the following holds: for all v such that $\zeta \leq v < \infty$,*

$$h_n(v) = v^{-\alpha_n}(1 + o(1)), \quad (2.2)$$

where $0 \leq \alpha_n = \alpha(\varepsilon) + o(1)$.

- (ii) *Let $0 < \delta < 1$. Then, for all v such that $r_n^{-\delta} \leq v \leq 1$ and all large enough n ,*

$$v^{-\alpha_n}(1 + o(1)) \leq h_n(v) \leq \frac{1}{1-\delta} v^{-\alpha_n(1-\frac{\delta}{2})}(1 + o(1)), \quad (2.3)$$

where α_n is as before.

Next, for $u \geq 0$ set

$$g_n(u) = r_n^{-1} G_n^{-1}(u/b_n). \quad (2.4)$$

It is plain that $g_n(v) = h_n^{-1}(v)$. It is plain also that both g_n and h_n are continuous monotone decreasing functions. The following lemma is tailored to deal with the case of extreme space scales. Recall that $\alpha \equiv \alpha(1)$.

Lemma 2.2: *Let r_n be an extreme space scale.*

(i) *For each fixed $u > 0$, for any sequence u_n such that $|u_n - u| \rightarrow 0$ as $n \rightarrow \infty$,*

$$g_n(u_n) \rightarrow u^{-(1/\alpha)}, \quad n \rightarrow \infty. \quad (2.5)$$

(ii) *There exists a constant $0 < C < \infty$ such that, for all n large enough,*

$$g_n(u) \leq u^{-1/\alpha} C, \quad 1 \leq u \leq b_n(1 - \Phi(1/(\beta\sqrt{n}))), \quad (2.6)$$

where Φ denotes the standard Gaussian distribution function.

The proofs of these two lemmata rely on Lemma 2.3 below. Denote by Φ and ϕ the standard Gaussian distribution function and density, respectively. Let B_n be defined through

$$b_n \frac{\phi(B_n)}{B_n} = 1, \quad (2.7)$$

and set $A_n = B_n^{-1}$

Lemma 2.3: *Let r_n be any space scale. Let \tilde{B}_n be a sequence such that, as $n \rightarrow \infty$,*

$$\delta_n := (\tilde{B}_n - B_n)/A_n \rightarrow 0. \quad (2.8)$$

Then, for all x such that $A_n x + \tilde{B}_n > 0$ for large enough n ,

$$b_n(1 - \Phi(A_n x + \tilde{B}_n)) = \frac{\exp(-x[1 + \frac{1}{2}A_n^2 x])}{1 + A_n^2 x} \{1 + \mathcal{O}(\delta_n[1 + A_n^2 + A_n^2 x]) + \mathcal{O}(A_n^2)\}. \quad (2.9)$$

Proof: The lemma is a direct consequence of the following expressions, valid for all $x > 0$,

$$\begin{aligned} 1 - \Phi(x) &= x^{-1}\phi(x) - r(x) \\ &= x^{-1}(1 - x^{-2})\phi(x) - s(x), \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} 0 < r(x) &= \int_x^\infty y^{-2}\phi(y)dy < x^{-3}\phi(x), \\ 0 < s(x) &= \int_x^\infty y^{-4}\phi(y)dy < x^{-5}\phi(x), \end{aligned} \quad (2.11)$$

(see [AS], p. 932). We leave the details to the reader. \square

We now prove Lemmata 2.1 and 2.2, beginning with Lemma 2.1.

Proof of Lemma 2.1: By definition of G_n we may write

$$h_n(v) = b_n \left(1 - \Phi(A_n \log(v^{\alpha_n}) + \bar{B}_n) \right), \quad (2.12)$$

where

$$\begin{aligned}\overline{B}_n &= (\beta\sqrt{n})^{-1} \log r_n, \\ \alpha_n &= (\beta\sqrt{n})^{-1} B_n.\end{aligned}\tag{2.13}$$

We first claim that for $v \geq r_n^{-1}$, which guarantees that $A_n \log(v^{\alpha_n}) + \overline{B}_n > 0$, the sequence \overline{B}_n satisfies the assumptions of Lemma 2.3. For this we use the known fact that the sequence \widehat{B}_n defined by

$$\widehat{B}_n = (2 \log b_n)^{\frac{1}{2}} - \frac{1}{2}(\log \log b_n + \log 4\pi)/(2 \log b_n)^{\frac{1}{2}},\tag{2.14}$$

satisfies

$$(\widehat{B}_n - B_n)/A_n = o(1)\tag{2.15}$$

(see [H], p. 434, paragraph containing Eq. (4)). By (2.10)-(2.11) we easily get that

$$b_n(1 - \Phi(\widehat{B}_n)) = 1 - (\log \log b_n)^2 (16 \log b_n)^{-1} (1 + o(1)),\tag{2.16}$$

whereas, by definition of b_n (see (1.10)),

$$b_n(1 - \Phi(\overline{B}_n)) = 1.\tag{2.17}$$

Since Φ is monotone and increasing, (2.16) and (2.17) imply that $\widehat{B}_n > \overline{B}_n$. Thus

$$(1 - \Phi(\overline{B}_n)) - (1 - \Phi(\widehat{B}_n)) = \Phi(\widehat{B}_n) - \Phi(\overline{B}_n) \geq \phi(\widehat{B}_n)(\widehat{B}_n - \overline{B}_n) \geq 0.\tag{2.18}$$

This, together with (2.16) and (2.17), yields

$$0 < \widehat{B}_n - \overline{B}_n < [b_n \phi(\widehat{B}_n)]^{-1} (\log \log b_n)^2 (16 \log b_n)^{-1} (1 + o(1)).\tag{2.19}$$

Now, by (2.7),

$$b_n \phi(\widehat{B}_n) = B_n [\phi(\widehat{B}_n)/\phi(B_n)] = B_n \exp\{-\frac{1}{2}(\widehat{B}_n - B_n)(\widehat{B}_n + B_n)\} \leq B_n(1 + o(1)),\tag{2.20}$$

where the final inequality follows from (2.15). Finally, combining (2.19) and (2.20) yields $0 < (\widehat{B}_n - \overline{B}_n)/A_n = o(1)$, and using again (2.15), we obtain $(\overline{B}_n - B_n)/A_n = o(1)$, which was the claim.

To control the behavior of the sequences A_n , α_n , and r_n , we will need an expression for the (of course well known) solution B_n of (2.7). Here is one ([Cr], p. 374):

$$B_n = (2 \log b_n)^{\frac{1}{2}} - \frac{1}{2}(\log \log b_n + \log 4\pi)/(2 \log b_n)^{\frac{1}{2}} + \mathcal{O}(1/\log b_n).\tag{2.21}$$

Note that so far we didn't make use of the assumption on r_n . The choice of r_n steps in only now: using (2.13), (2.21), the fact that $2 \log b_n = (2 \log 2)m_n = \beta_c^2(m_n/n)n$, and the just established fact that $(\overline{B}_n - B_n)/A_n \rightarrow 0$, we obtain, for intermediate space scales,

$$\begin{aligned}\log r_n &= \beta\beta_c(\varepsilon)n(1 + o(1)), \\ \log b_n &= \frac{1}{2}\beta_c^2(\varepsilon)n(1 + o(1)), \\ \alpha_n &= \alpha(\varepsilon)(1 + o(1)),\end{aligned}\tag{2.22}$$

and, for short space scales,

$$\begin{aligned}\log r_n &= o(n), \\ \log b_n &= o(n), \\ \alpha_n &= o(1).\end{aligned}\tag{2.23}$$

Finally for extreme space scales, wrting $\beta_c(1) \equiv \beta_c$, we have that $2 \log b_n = (2 \log 2)m_n = \beta_c^2 n(1 - C/n)$ for some constant $0 < C < \infty$. Thus, instead of (2.22), we get:

$$\begin{aligned}\log b_n &= \frac{1}{2}\beta_c^2 n(1 - C/n), \\ \log r_n &= \beta\beta_c n(1 - o(1)), \\ \alpha_n &\leq \alpha \quad \text{and} \quad \alpha_n = \alpha(1 - o(1)).\end{aligned}\tag{2.24}$$

We are now equipped to prove Lemma 2.1. By Lemma 2.3, for all $v \geq r_n^{-1}$,

$$h_n(v) = \frac{\exp(-\alpha_n \log v [1 + \frac{1}{2}A_n^2 \alpha_n \log v])}{1 + A_n^2 \alpha_n \log v} \{1 + \mathcal{O}(\delta_n[1 + A_n^2 + A_n^2 \alpha_n \log v]) + \mathcal{O}(A_n^2)\},\tag{2.25}$$

where $\delta_n \downarrow 0$ as $n \uparrow \infty$. Therefore, for each fixed $0 < v < \infty$, and all large enough n so that $v > r_n^{-1}$,

$$h_n(v) = v^{-\alpha_n}(1 + o(1)).\tag{2.26}$$

This together with (2.22) and (2.23) proves assertion (i) of the lemma. To prove assertion (ii) note that by (2.13), since $A_n = B_n^{-1}$, $A_n^2 \alpha_n = \frac{1}{\log r_n} \frac{\overline{B}_n}{B_n}$ where $\frac{\overline{B}_n}{B_n} = 1 + o(1)$ (see the paragraph following (2.20)). Thus, for all v satisfying $r_n^{-\delta} \leq v \leq 1$, we have

$$-\delta \frac{\overline{B}_n}{B_n} \leq A_n^2 \alpha_n \log v \leq 0.\tag{2.27}$$

Combining this and (2.25) immediately yields the bounds (2.3). The proof of Lemma 2.1 is now done. \square

Proof of Lemma 2.2: Up until (2.25) we proceed exactly as in the proof of Lemma 2.1. Now, by (2.25), for each fixed $0 \leq v < \infty$, any sequence v_n such that $|v_n - v| \rightarrow 0$, and all large enough n (so that $v > r_n^{-1}$),

$$h_n(v_n) = v_n^{-\alpha_n}(1 + o(1)) = v^{-\alpha(1-o(1))}(1 + o(1)).\tag{2.28}$$

This and the relation $g_n(v) = h_n^{-1}(v)$ imply that for each fixed $0 < u < \infty$, any sequence u_n such that $|u_n - u| \rightarrow 0$, and all large enough n (so that $u < h_n(r_n^{-1})$),

$$g_n(u_n) = u_n^{-(1/\alpha_n)}(1 + o(1)) = u^{-(1/\alpha)(1+o(1))}(1 + o(1)),\tag{2.29}$$

which is tantamount to assertion (i) of the lemma.

To prove assertion (ii) assume that $r_n^{-1} \leq v \leq 1$. Recall that h_n is a monotonous function so that if $h_n(v) = g_n^{-1}(v)$ for all $r_n^{-1} \leq v \leq 1$, then $g_n(u) = h_n^{-1}(u)$ for all $h_n(1) \leq u \leq h_n(r_n^{-1})$. Now $h_n(1) = b_n G_n(r_n) = 1$, as follows from (1.10), and $h_n(r_n^{-1}) = b_n G_n(1) =$

$b_n(1 - \Phi(1/(\beta\sqrt{n})))$. Observe next that $r_n^{-1} \leq v \leq 1$ is equivalent to $-1 \leq A_n^2 \log v^{\alpha_n} \leq 0$. Therefore, by (2.25), for large enough n ,

$$h_n(v) \geq (1 - 2\delta_n)v^{-\alpha_n}, \quad r_n^{-1} \leq v \leq 1. \quad (2.30)$$

By monotonicity of h_n ,

$$g_n(u) = h_n^{-1}(u) \leq (1 - 2\delta_n)^{1/\alpha_n} u^{-1/\alpha_n}, \quad 1 \leq u \leq b_n(1 - \Phi(1/(\beta\sqrt{n}))). \quad (2.31)$$

From this and the fact that $\alpha_n \leq \alpha$ (see (2.24)), (2.6) readily obtains. This concludes the proof of the lemma. \square

Remark: We see from the proof of Lemma 2.2 that the lemma holds true not only for extreme scales, but for intermediate scales also provided one replaces α by $\alpha(\varepsilon)$ everywhere.

3. The jump chain: some estimates.

This section is about the jump chain J_n , i.e. the simple random walk. We gather here all the results that will be needed later to reduce Condition (A1) and Condition (A2) of Theorem 1.8 to conditions that are independent from J_n . Proposition 3.1 below is central to this scheme. It will allow us to substitute the stationary chain for the jump chain after $\theta_n \sim n^2$ steps have been taken.

Proposition 3.1: *Set $\theta_n = 2 \lceil \frac{3}{2}(n-1) \log 2 / |\log(1 - \frac{2}{n})| \rceil$. For all pairs $x \in \mathcal{V}_n, y \in \mathcal{V}_n$, and all $i \geq 0$,*

$$P_{\pi_n}(J_n(i + \theta_n) = y, J_n(0) = x) + P_{\pi_n}(J_n(i + 1 + \theta_n) = y, J_n(0) = x) = 2(1 + \delta_n)\pi_n(x)\pi_n(y), \quad (3.1)$$

where $|\delta_n| \leq 2^{-n}$.

The next two propositions are technical estimates needed in the proofs of Proposition 4.1 and Proposition 6.4 respectively. Let $p_n^l(\cdot, \cdot)$ denote the l steps transition probabilities of J_n , and let $\text{dist}(x, y) := \#\{i \in \{1, \dots, n\} : x_i \neq y_i\}$ be the Hamming distance.

Proposition 3.2: *For all $m \leq n^2$,*

$$\sum_{l=1}^{2m} p_n^{l+2}(z, z) \leq \frac{c}{n^2}, \quad \forall z \in \mathcal{V}_n, \quad (3.2)$$

for some constant $0 < c < \infty$.

Proposition 3.3: *For all $m \leq n^2$, for all pairs of distinct vertices $y, z \in \mathcal{V}_n$ satisfying $\text{dist}(y, z) = \frac{n}{2}(1 - o(1))$,*

$$\sum_{l=1}^{2m} p_n^{l+2}(y, z) \leq e^{-cn}, \quad (3.3)$$

for some constant $0 < c < \infty$.

We first prove Proposition 3.1. For this we will use the following classical bound by Diaconis and Stroock [DS] for the total variation distance to stationarity of reversible Markov

chains. Recall that the total variation distance between two probabilities μ and μ' on \mathcal{V}_n° is defined as

$$\|\mu - \mu'\|_{TV} = \max_{A \subset \mathcal{V}_n^\circ} |\mu(A) - \mu'(A)| = \frac{1}{2} \sum_{x \in \mathcal{V}_n^\circ} |\mu(x) - \mu'(x)|.$$

Proposition 3.4: [[DS], Proposition 3] *Let ν , $Q(x, y)$ be a reversible Markov chain on a finite set X . Assume that the one-step transition probability matrix Q is irreducible with eigenvalues $1 = \beta_0 > \beta_1 \geq \dots \geq \beta_{k-1} \geq -1$. Then, for all $x \in X$ and $m \in \mathbb{N}$,*

$$4\|Q^m(x, \cdot) - \nu(\cdot)\|_{TV}^2 \leq \frac{1 - \nu(x)}{\nu(x)} \beta_*^{2m}, \quad \beta_* = \min(\beta_1, |\beta_{k-1}|). \quad (3.4)$$

Proposition 3.4 cannot be applied directly to the jump chain J_n , since it is periodic with period two, but it can be applied to the aperiodic chains obtained by observing J_n at even, respectively, odd times. In view of doing this let us partition the cube into the sub-cubes $\mathcal{V}_n = \mathcal{V}_n^{od} \cup \mathcal{V}_n^{ev}$ of vertices that are at odd, respectively, even distance of the vertex $x = (1, 1, \dots, 1)$:

$$\begin{aligned} \mathcal{V}_n^{od} &= \{x \in \mathcal{V}_n \mid \sum_{i=1}^n (x_i + 1)/2 \text{ is odd}\}, \\ \mathcal{V}_n^{ev} &= \{x \in \mathcal{V}_n \mid \sum_{i=1}^n (x_i + 1)/2 \text{ is even}\}. \end{aligned} \quad (3.5)$$

To each of these sub-cubes we associate a chain, J_n^{od} and J_n^{ev} , as follows. Let the symbol $^\circ$ denote either *od* or *ev*. Set $\mathcal{E}_n^\circ = \{(x, y) \in \mathcal{V}_n^\circ \times \mathcal{V}_n^\circ : \sum_{i=1}^n |x_i - y_i| = 4\}$. Then $(J_n^\circ(k), k \in \mathbb{N})$ is the chain on \mathcal{V}_n° with transition probabilities $p_n^\circ(x, y) = P^\circ(J_n^\circ(i+1) = y \mid J_n^\circ(i) = x) = P_{\pi_n}(J_n(i+2) = y \mid J_n(i) = x)$, that is,

$$p_n^\circ(x, y) = \begin{cases} \frac{2}{n^2} & \text{if } (x, y) \in \mathcal{E}_n^\circ, \\ \frac{1}{n} & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

Clearly J_n° is aperiodic and has a unique reversible invariant measure π_n° given by

$$\pi_n^\circ(x) = \begin{cases} 2^{-n+1} & \text{if } x \in \mathcal{V}_n^\circ, \\ 0, & \text{otherwise.} \end{cases} \quad (3.7)$$

In what follows we denote by P° the law of J_n° with initial distribution π_n° .

Applying Proposition 3.4 to each of the two chains J_n° yields:

Lemma 3.5: *Let θ_n and δ_n be as in Proposition 3.1. Then, for all $x \in \mathcal{V}_n^\circ$, all $y \in \mathcal{V}_n$, and large enough n , $P^\circ(J_n^\circ(l) = y \mid J_n^\circ(0) = x) = (1 + \delta_n)\pi_n^\circ(y)$, for all $l \geq \theta_n/2$.*

Proof of Lemma 3.5: The eigenvalues of the transition matrix $Q^\circ := (p_n^\circ(x, y))_{\mathcal{V}_n^\circ \times \mathcal{V}_n^\circ}$ of J_n° are $(1 - 2\frac{j}{n})^2$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, where $^\circ$ denotes either *od* or *ev*. The proof of this statement uses the following three facts: (i) firstly, the eigenvalues of the transition matrix $Q := (p_n(x, y))_{\mathcal{V}_n \times \mathcal{V}_n}$ of J_n are $1 - 2j/n$, $0 \leq j \leq n$ (see, for example, [DS] example 2.2 p. 45); (ii) next, by definition of Q° , $Q^2 = Q^{ev} + Q^{od}$ and $Q^{ev}Q^{od} = Q^{od}Q^{ev} = 0$; (iii) finally,

Q^{ev} and Q^{od} can be obtained from one another by permutation of their rows and columns. Now it follows from (iii) that Q^{ev} and Q^{od} must have the same spectrum. This fact combined with (i) and (ii) imply that this spectrum coincide with that of Q^2 . The conclusion then follows from (i).

We may thus apply (3.4) to the chain J_n° with $\beta_* = (1 - \frac{2}{n})^2$. Choosing $m = \theta_n/2 = \lceil \frac{3}{2}(n-1) \log 2 / |\log(1 - \frac{2}{n})| \rceil$, this yields $P^\circ(J_n^\circ(l) = y \mid J_n^\circ(0) = x) = (1 + \delta_n)\pi_n^\circ(y)$ where $\delta_n^2 \leq \frac{1}{4}2^{3(n-1)}(1 - \frac{2}{n})^{2\theta_n} \leq 2^{-3n+1}$ for all n large enough, and thus $|\delta_n| \leq 2^{-n}$. The lemma is proven. \square

Proof of Proposition 3.1: Proposition 3.1 is an immediate consequence of Lemma 3.5. To see this set (to simplify the notation we drop the dependence of P on its initial distribution π_n in this proof) $\Delta = P(J_n(i + \theta_n) = y, J_n(0) = x) + P(J_n(i + 1 + \theta_n) = y, J_n(0) = x)$. Since J_n is started from its invariant measure π_n , $\Delta = P(J_n(i + \theta_n) = y \mid J_n(0) = x)\pi_n(x) + P(J_n(i + 1 + \theta_n) = y \mid J_n(0) = x)\pi_n(x)$. Without loss of generality we may assume that $i + \theta_n$ is even, and set $i + \theta_n = 2l$. Then, using the notation $x \sim y \Leftrightarrow \text{dist}(x, y) = 1$, $\Delta/\pi_n(x)$ can be rewritten as

$$\begin{aligned} \frac{\Delta}{\pi_n(x)} &= P(J_n(2l) = y \mid J_n(0) = x) + P(J_n(2l + 1) = y \mid J_n(0) = x) \\ &= P(J_n(2l) = y \mid J_n(0) = x) + \frac{1}{n} \sum_{z \sim x} P(J_n(2l + 1) = y \mid J_n(1) = z) \\ &= P(J_n^{ev}(l) = y \mid J_n^{ev}(0) = x) \mathbb{I}_{\{x \in \mathcal{V}_n^{ev}\}} + P(J_n^{od}(l) = y \mid J_n^{od}(0) = x) \mathbb{I}_{\{x \in \mathcal{V}_n^{od}\}} \\ &\quad + \frac{1}{n} \sum_{z \sim x} \left[P(J_n^{od}(l) = y \mid J_n^{od}(1) = z) \mathbb{I}_{\{x \in \mathcal{V}_n^{ev}\}} + P(J_n^{ev}(l) = y \mid J_n^{ev}(1) = z) \mathbb{I}_{\{x \in \mathcal{V}_n^{od}\}} \right], \end{aligned} \quad (3.8)$$

and, making use of Lemma 3.5,

$$\frac{\Delta}{\pi_n(x)} = (1 + \delta_n) \left[\pi_n^{ev}(y) \mathbb{I}_{\{x \in \mathcal{V}_n^{ev}\}} + \pi_n^{od}(y) \mathbb{I}_{\{x \in \mathcal{V}_n^{od}\}} + \pi_n^{od}(y) \mathbb{I}_{\{x \in \mathcal{V}_n^{ev}\}} + \pi_n^{ev}(y) \mathbb{I}_{\{x \in \mathcal{V}_n^{od}\}} \right]. \quad (3.9)$$

Now, exactly one of the indicator function in the right hand side of (3.8) is non zero, so that, by (3.7) and (1.9), $\Delta = 2(1 + \delta_n)\pi_n(x)\pi_n(y)$. The proof of Proposition 3.1 is done. \square

We now prove Proposition 3.2 and Proposition 3.3.

Proof of Proposition 3.2: Consider the Ehrenfest chain on state space $\{0, \dots, 2n\}$ with one step transition probabilities $r_n(i, i + 1) = \frac{i}{2n}$ and $r_n(i, i - 1) = 1 - \frac{i}{2n}$. Denote by $r_n^l(\cdot, \cdot)$ its l steps transition probabilities. It is well known (see e.g. [BG]) that $p_n^l(z, z) = r_n^l(0, 0)$ for all $l \geq 0$ and all $z \in \mathcal{V}_n$. Hence $\sum_{l=1}^{2m} p_n^{l+2}(z, z) = \sum_{l=1}^{2m} r_n^{l+2}(0, 0)$. It is in turn well known (see [Kem], p. 25, formula (4.18)) that

$$r_n^l(0, 0) = 2^{-n} \sum_{k=0}^{2n} \binom{2n}{l} \left(1 - \frac{k}{n}\right)^l, \quad l \geq 1. \quad (3.10)$$

Note that by symmetry, $r_n^{2l+1}(0, 0) = 0$. Simple calculations yield $r_n^4(0, 0) = \frac{c_2}{n^2}$, $r_n^6(0, 0) = \frac{c_3}{n^3}$, and $r_n^8(0, 0) = \frac{c_4}{n^4}$, for some constants $0 < c_i < \infty$, $1 \leq i \leq 3$. Thus, if $m \leq 3$,

$\sum_{l=1}^{2m} r_n^{l+2}(0,0) \leq \frac{c}{n^2}$ for some constant $0 < c < \infty$. If now $m > 3$, write $\sum_{l=1}^{2m} r_n^{l+2}(0,0) = r_n^4(0,0) + r_n^6(0,0) + \sum_{l=6}^{2m} r_n^{l+2}(0,0)$, and use that by (3.10),

$$\sum_{l=6}^{2m} r_n^{l+2}(0,0) = 2^{-n} \sum_{k=0}^{2n} \binom{2n}{l} \sum_{l=6}^{2m} \left(1 - \frac{k}{n}\right)^{l+2} \leq 2^{-n} \sum_{k=0}^{2n} \binom{2n}{l} \left(1 - \frac{k}{n}\right)^8 \sum_{j=0}^{m-1} \left(1 - \frac{k}{n}\right)^j. \quad (3.11)$$

Since $|1 - \frac{k}{n}| \leq 1$, $\sum_{l=6}^{2m} r_n^{l+2}(0,0) \leq 2^{-n} \sum_{k=0}^{2n} \binom{2n}{l} \left(1 - \frac{k}{n}\right)^8 m = m r_n^8(0,0) \leq n^2 \frac{c_4}{n^4}$, so that $\sum_{l=1}^{2m} r_n^{l+2}(0,0) \leq \frac{c_2}{n^2} + \frac{c_3}{n^3} + n^2 \frac{c_4}{n^4} \leq \frac{c}{n^2}$ for some constant $0 < c < \infty$. The lemma is proven. \square

Proof of Proposition 3.3: This estimate is proved using a d -dimensional version of the Ehrenfest scheme known as the lumping procedure, and studied e.g. in [BG]. In what follows we mostly stick to the notations of [BG], hoping that this will create no confusion. Without loss of generality we may take $y \equiv 1$ to be the vertex whose coordinates all take the value 1. Let γ^Λ be the map (1.7) of [BG] derived from the partition of $\Lambda \equiv \{1, \dots, n\}$ into $d = 2$ classes, $\Lambda = \Lambda_1 \cup \Lambda_2$, defined through the relation: $i \in \Lambda_1$ if the i^{th} coordinate of z is 1, and $i \in \Lambda_2$ otherwise. The resulting lumped chain $X_n^\Lambda \equiv \gamma^\Lambda(J_n)$ has range $\Gamma_{n,2} = \gamma^\Lambda(\mathcal{V}_n) \subset [-1, 1]^2$. Note that the vertices 1 and y of \mathcal{V}_n are mapped respectively on the corners $1 \equiv (1, 1)$ and $x \equiv (1, -1)$ of $[-1, 1]^2$. Without loss of generality we may assume that $0 \in \Gamma_{n,2}$. Now, denoting by \mathbb{P}° the law of X_n^Λ , we have, $p_n^{l+2}(y, z) = \mathbb{P}^\circ(X_n^\Lambda(l+2) = x \mid X_n^\Lambda(0) = 1)$. Let $\tau_x^{x'} = \inf\{k > 0 \mid X_n^\Lambda(0) = x', X_n^\Lambda(k) = x\}$. Starting from 1, the lumped chain may visit 0 before it visits x or not. Thus $p_n^{l+2}(1, z) = \mathbb{P}^\circ(X_n^\Lambda(l+2) = x, \tau_0^1 < \tau_x^1) + \mathbb{P}^\circ(X_n^\Lambda(l+2) = x, \tau_0^1 \geq \tau_x^1)$. On the one hand, using Theorem 3.2 of [BG], $\mathbb{P}^\circ(X_n^\Lambda(l+2) = x, \tau_0^1 \geq \tau_x^1) \leq \mathbb{P}^\circ(\tau_x^1 \leq \tau_0^1) \leq e^{-c_1 n}$ for some constant $0 < c_1 < \infty$. On the other hand, conditioning on the time of the last return to 0 before time $l+2$, and bounding the probability of the latter event by 1, we readily get

$$\mathbb{P}^\circ(X_n^\Lambda(l+2) = x, \tau_0^1 < \tau_x^1) \leq (l+2) \mathbb{P}^\circ(\tau_x^0 < \tau_0^0) = (l+2) \frac{\mathbb{Q}_n(x)}{\mathbb{Q}_n(0)} \mathbb{P}^\circ(\tau_0^x < \tau_x^x), \quad (3.12)$$

where the last line follows from reversibility, and where \mathbb{Q}_n , defined in Lemma 2.2 of [BG], denotes the invariant measure of X_n^Λ . Since $\mathbb{P}^\circ(\tau_0^x < \tau_x^x) \leq 1$ we are left to estimate the ratio of invariant masses in (3.12). From the assumption that $\text{dist}(y, z) = \frac{n}{2}(1 - o(1))$, it follows that $\Lambda_1 = n - \Lambda_2 = \frac{n}{2}(1 - o(1))$. Therefore, by (2.4) of [BG], $\frac{\mathbb{Q}_n(x)}{\mathbb{Q}_n(0)} \leq e^{-c_2 n}$ for some constant $0 < c_2 < \infty$. Gathering our bounds we arrive at $p_n^{l+2}(1, z) \leq e^{-c_1 n} + (l+2)e^{-c_2 n}$, which proves the claim of the lemma. \square

4. Preparations to the verification of Conditions (A1) and (A2).

We now capitalize on the estimates of Section 3 and, as a first step towards the verification of Conditions (A1) and (A2), prove that these conditions can be replaced by simple ones, where all quantities depending on the jump chain have been averaged out. To state our central result we need a little notation. Set $k_n(t) := \lfloor a_n t \rfloor$. Let $\pi_n^{J,t}(x)$ denote the average number of visits of J_n to x during the first $k_n(t)$ steps,

$$\pi_n^{J,t}(x) = k_n^{-1}(t) \sum_{j=1}^{k_n(t)} \mathbb{1}_{\{J_n(j-1)=x\}}, \quad x \in \mathcal{V}_n. \quad (4.1)$$

For $y \in \mathcal{V}_n$ and $u > 0$ further set

$$h_n^u(y) = \sum_{x \in \mathcal{V}_n} p_n(y, x) \exp\{-uc_n \lambda_n(x)\}, \quad (4.2)$$

and define

$$\begin{aligned} \nu_n^{J,t}(u, \infty) &= \sum_{j=1}^{k_n(t)} h_n^u(J_n(j-1)) = k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{J,t}(y) h_n^u(y), \\ (\sigma_n^{J,t})^2(u, \infty) &= \sum_{j=1}^{k_n(t)} (h_n^u(J_n(j-1)))^2 = k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{J,t}(y) (h_n^u(y))^2. \end{aligned} \quad (4.3)$$

By assumption, the initial distribution, μ_n , is the invariant measure π_n of J_n . This implies that the chain variables $(J_n(j), j \geq 1)$ satisfy

$$P_{\pi_n}(J_n(j) = x) = \pi_n(x) = 2^{-n} \quad \text{for all } x \in \mathcal{V}_n, \text{ and all } j \geq 1. \quad (4.4)$$

Hence

$$\begin{aligned} E_{\pi_n} [\pi_n^{J,t}(y)] &= \pi_n(y), \\ E_{\pi_n} [\nu_n^{J,t}(u, \infty)] &= \frac{k_n(t)}{a_n} \nu_n(u, \infty), \\ E_{\pi_n} [(\sigma_n^{J,t})^2(u, \infty)] &= \frac{k_n(t)}{a_n} \sigma_n^2(u, \infty), \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \nu_n(u, \infty) &= a_n \sum_{x \in \mathcal{V}_n} \pi_n(x) h_n^u(x), \\ \sigma_n^2(u, \infty) &= a_n \sum_{x \in \mathcal{V}_n} \pi_n(x) (h_n^u(x))^2. \end{aligned} \quad (4.6)$$

We will often refer to Proposition 4.1 below as to an ergodic theorem.

Proposition 4.1: *Let $\rho_n > 0$ be a decreasing sequence satisfying $\rho_n \downarrow 0$ as $n \uparrow \infty$. There exists a sequence of subsets $\Omega_{n,0}^\tau \subset \Omega^\tau$ with $\mathbb{P}((\Omega_{n,0}^\tau)^c) < \frac{\theta_n k_n(t)}{\rho_n a_n^2}$, and such that, on $\Omega_{n,0}^\tau$, the following holds: for all $t > 0$ and all $u > 0$,*

$$P_{\pi_n} (|\nu_n^{J,t}(u, \infty) - E_{\pi_n} [\nu_n^{J,t}(u, \infty)]| \geq \epsilon) \leq \epsilon^{-2} \Theta_n(t, u), \quad \forall \epsilon > 0, \quad (4.7)$$

where, for some constant $0 < c < \infty$,

$$\Theta_n(t, u) = \left(\frac{k_n(t)}{a_n} \right)^2 \frac{\nu_n^2(u, \infty)}{2^n} + \frac{k_n(t)}{a_n} \sigma_n^2(u, \infty) + c \frac{\nu_n(2u, \infty)}{n^2} + \rho_n [\mathbb{E} \nu_n(u, \infty)]^2. \quad (4.8)$$

In addition, for all $t > 0$ and all $u > 0$,

$$P_{\pi_n} ((\sigma_n^{J,t})^2(u, \infty) \geq \epsilon') \leq \frac{k_n(t)}{\epsilon' a_n} \sigma_n^2(u, \infty), \quad \forall \epsilon' > 0. \quad (4.9)$$

Remark: Clearly that Proposition 4.1 will be useful only if $a_n \gg \theta_n$, namely, if the number of steps $k_n(t)$ taken by the jump chain is large compared to its stationary time. From the relations $a_n \sim b_n$ and (1.10) we see that if a_n is bounded below then so is r_n . This explains our lower bound on r_n in Assertion (i) of Theorem 1.2.

Remark: The small simplification that the choice $\mu_n = \pi_n$ introduces is that $E_{\mu_n} [\pi_n^{J,t}(y)]$ exactly is the invariant mass $\pi_n(y)$, and this in turn yields exact expressions for $E_{\mu_n} [\nu_n^{J,t}(u, \infty)]$ and $E_{\mu_n} [(\sigma_n^{J,t})^2(u, \infty)]$ in (4.5). When μ_n is not the invariant measure of the jump chain one uses first Lemma 3.5 to approximate $E_{\mu_n} [\pi_n^{J,t}(y)]$ by its invariant mass. Proposition 4.1 is then proved along the same lines. Choosing π_n for initial distribution is generic inasmuch as the error introduced by this approximation is typically negligible.

Remark: A slightly different version of Proposition 4.1 will be needed to deal with the case of extreme scales, where a representation of the landscape will be substituted for the original one. See Proposition 6.4 of Section 6.2.

Proof of Proposition 4.1: The upper bound (4.9) plainly results from a first order Tchebychev inequality and the expression (4.5) of $E_{\pi_n} [(\sigma_n^{J,t})^2(u, \infty)]$. The proof of (4.7) is a little more involved. Using a second order Tchebychev inequality together with the expressions (4.5) and (4.6) of $E_{\pi_n} [\nu_n^{J,t}(u, \infty)]$ and $\nu_n(u, \infty)$, we get,

$$P_{\pi_n} (|\nu_n^{J,t}(u, \infty) - E_{\pi_n} [\nu_n^{J,t}(u, \infty)]| \geq \epsilon) \leq \epsilon^{-2} E_{\pi_n} \left[k_n(t) \sum_{y \in \mathcal{V}_n} (\pi_n^{J,t}(y) - \pi_n(y)) h_n^u(y) \right]^2. \quad (4.10)$$

Expanding the r.h.s. of (4.10) yields

$$\epsilon^{-2} k_n^2(t) \sum_{x \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n} h_n^u(x) h_n^u(y) E_{\pi_n} (\pi_n^{J,t}(x) - \pi_n(x)) (\pi_n^{J,t}(y) - \pi_n(y)). \quad (4.11)$$

In view of (4.1), setting $\Delta_{ij}(x, y) = P_{\pi_n} (J_n(i-1) = x, J_n(j-1) = y) - \pi_n(x)\pi_n(y)$, the expectation in (4.11) may be expressed as

$$k_n^2(t) E_{\pi_n} (\pi_n^{J,t}(x) - \pi_n(x)) (\pi_n^{J,t}(y) - \pi_n(y)) = \sum_{i=1}^{k_n(t)} \sum_{j=1}^{k_n(t)} \Delta_{ij}(x, y). \quad (4.12)$$

For θ_n defined as in Proposition 3.1, we now break the sum in the r.h.s. of (4.12) into three terms:

$$\begin{aligned} (\bar{I}) &= 2 \sum_{1 \leq i \leq k_n(t)} \sum_{i+\theta_n \leq j \leq k_n(t)} \Delta_{ij}(x, y), \\ (\bar{II}) &= \sum_{1 \leq i \leq k_n(t)} \mathbb{1}_{\{i=j\}} \Delta_{ij}(x, y), \\ (\bar{III}) &= 2 \sum_{1 \leq i \leq k_n(t)} \sum_{i < j < i+\theta_n} \Delta_{ij}(x, y). \end{aligned} \quad (4.13)$$

Consider first (\bar{I}) . By Proposition 3.1,

$$\begin{aligned}
(\bar{I}) &\leq 2 \sum_{1 \leq i \leq k_n(t)} \sum_{\lfloor (i+\theta_n)/2 \rfloor \leq l \leq \lceil k_n(t)/2 \rceil} [\Delta_{i(2l)}(x, y) + \Delta_{i(2l+1)}(x, y)] \\
&\leq 2 \sum_{1 \leq i \leq k_n(t)} \sum_{\lfloor (i+\theta_n)/2 \rfloor \leq l \leq \lceil k_n(t)/2 \rceil} 2|\delta_n| \pi_n(x) \pi_n(y) \\
&\leq |\delta_n| k_n^2(t) \pi_n(x) \pi_n(y).
\end{aligned} \tag{4.14}$$

where $|\delta_n| \leq 2^{-n}$. Turning to the term (\bar{II}) , we have,

$$\begin{aligned}
(\bar{II}) &= \sum_{1 \leq i \leq k_n(t)} \Delta_{ii}(x, x) \mathbb{I}_{\{x=y\}} \\
&= k_n(t) [P_{\pi_n}(J_n(i-1) = x) - \pi_n^2(x)] \mathbb{I}_{\{x=y\}} \\
&= k_n(t) \pi_n(x) (1 - \pi_n(x)) \mathbb{I}_{\{x=y\}},
\end{aligned} \tag{4.15}$$

where the last equality follows from (4.4). Finally,

$$\begin{aligned}
(\bar{III}) &\leq 2 \sum_{i=1}^{k_n(t)} \sum_{l=1}^{\theta_n-1} P_{\pi_n}(J_n(i-1) = x, J_n(i+l-1) = y) \\
&\leq 2 \sum_{i=1}^{k_n(t)} \sum_{l=1}^{\theta_n-1} P_{\pi_n}(J_n(i-1) = x) P_{\pi_n}(J_n(i+l-1) = y \mid J_n(i-1) = x) \\
&= 2k_n(t) \pi_n(x) \sum_{l=1}^{\theta_n-1} p_n^l(x, y),
\end{aligned} \tag{4.16}$$

where $p_n^l(\cdot, \cdot)$ denote the l -steps transition matrix of J_n . Combining our bounds on (\bar{I}) , (\bar{II}) , and (\bar{III}) with (4.11) we get that, for all $\epsilon > 0$,

$$P_{\pi_n}(|\nu_n^{J,t}(u, \infty) - E_{\pi_n}[\nu_n^{J,t}(u, \infty)]| \geq \epsilon) \leq \epsilon^{-2}[(I) + (II) + (III)], \tag{4.17}$$

where

$$\begin{aligned}
(I) &= |\delta_n| k_n^2(t) \sum_{x \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n} h_n^u(x) h_n^u(y) \pi_n(x) \pi_n(y), \\
(II) &= k_n(t) \sum_{x \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n} h_n^u(x) h_n^u(y) \pi_n(x) (1 - \pi_n(x)) \mathbb{I}_{\{x=y\}}, \\
(III) &= 2k_n(t) \sum_{x \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n} h_n^u(x) h_n^u(y) \pi_n(x) \sum_{l=1}^{\theta_n-1} p_n^l(x, y).
\end{aligned} \tag{4.18}$$

By (4.6),

$$\begin{aligned}
(I) &\leq \left(\frac{k_n(t)}{a_n} \right)^2 \frac{\nu_n^2(u, \infty)}{2^n}, \\
(II) &\leq \frac{k_n(t)}{a_n} \sigma_n^2(u, \infty).
\end{aligned} \tag{4.19}$$

To further express the third term in (4.18) note that, by (4.2),

$$\sum_{y \in \mathcal{V}_n} p_n^l(x, y) h_n^u(y) = \sum_{y \in \mathcal{V}_n} p_n^l(x, y) \sum_{z \in \mathcal{V}_n} p_n(y, z) e^{-uc_n \lambda_n(z)} = \sum_{z \in \mathcal{V}_n} p_n^{l+1}(x, z) e^{-uc_n \lambda_n(z)}, \quad (4.20)$$

and,

$$\begin{aligned} \sum_{x \in \mathcal{V}_n} \pi_n(x) h_n^u(x) p_n^{l+1}(x, z) &= \sum_{y \in \mathcal{V}_n} e^{-uc_n \lambda_n(y)} \sum_{x \in \mathcal{V}_n} \pi_n(x) p_n(x, y) p_n^{l+1}(x, z) \\ &= \sum_{y \in \mathcal{V}_n} e^{-uc_n \lambda_n(y)} \pi_n(y) p_n^{l+2}(y, z), \end{aligned} \quad (4.21)$$

where the last equality follows by reversibility. Hence,

$$\begin{aligned} (III) &= 2k_n(t) \sum_{l=1}^{\theta_n-1} \sum_{z \in \mathcal{V}_n} \left[\sum_{x \in \mathcal{V}_n} \pi_n(x) h_n^u(x) p_n^{l+1}(x, z) \right] e^{-uc_n \lambda_n(z)}, \\ &= 2 \sum_{l=1}^{\theta_n-1} k_n(t) \sum_{z \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n} \pi_n(y) e^{-uc_n(\lambda_n(y) + \lambda_n(z))} p_n^{l+2}(y, z) \\ &= 2 \sum_{l=1}^{\theta_n-1} [(III)_{1,l} + (III)_{2,l}]. \end{aligned} \quad (4.22)$$

where, distinguishing the cases $z = y$ and $z \neq y$,

$$\begin{aligned} (III)_{1,l} &= \sum_{z \in \mathcal{V}_n} k_n(t) \pi_n(z) e^{-2uc_n \lambda_n(z)} p_n^{l+2}(z, z), \\ (III)_{2,l} &= \sum_{z \in \mathcal{V}_n} \sum_{y \in \mathcal{V}_n: y \neq z} k_n(t) \pi_n(y) e^{-uc_n(\lambda_n(y) + \lambda_n(z))} p_n^{l+2}(y, z). \end{aligned} \quad (4.23)$$

One easily checks that $\theta_n \leq 2m$ with $m \leq n^2$. Thus, by Proposition 3.2,

$$\sum_{l=1}^{\theta_n-1} (III)_{1,l} = \sum_{z \in \mathcal{V}_n} k_n(t) \pi_n(z) e^{-2uc_n \lambda_n(z)} \sum_{l=1}^{\theta_n-1} p_n^{l+2}(z, z) \leq cn^{-2} \nu_n(2u, \infty). \quad (4.24)$$

for some constant $0 < c < \infty$.

The next lemma is designed to deal with the second sum in the last line of (4.22).

Lemma 4.2: *Let $\rho_n > 0$ be a decreasing sequence satisfying $\rho_n \downarrow 0$ as $n \uparrow \infty$. There exists a sequence of subsets $\Omega_{n,0}^\tau \subset \Omega^\tau$ with $\mathbb{P}((\Omega_{n,0}^\tau)^c) < \frac{\theta_n k_n(t)}{\rho_n a_n^2}$, and such that, on $\Omega_{n,0}^\tau$,*

$$\sum_{l=1}^{\theta_n-1} (III)_{2,l} < \rho_n [\mathbb{E} \nu_n(u, \infty)]^2. \quad (4.25)$$

Proof: By a first order Tchebychev inequality, for all $\eta > 0$, $\mathbb{P}\left(\sum_{l=1}^{\theta_n-1} (III)_{2,l} \geq \eta\right) \leq \eta^{-1} \sum_{l=1}^{\theta_n-1} \mathbb{E}(III)_{2,l}$. Next, for all $y \neq z \in \mathcal{V}_n \times \mathcal{V}_n$, by independence, $\mathbb{E}\left[e^{-uc_n(\lambda_n(y) + \lambda_n(z))}\right] = \left[a_n^{-1} \mathbb{E}\nu_n(u, \infty)\right]^2$. Thus,

$$\sum_{l=1}^{\theta_n-1} \mathbb{E}(III)_{2,l} \leq \frac{k_n(t)}{a_n^2} [\mathbb{E}\nu_n(u, \infty)]^2 \sum_{l=1}^{\theta_n-1} \sum_{z \in \mathcal{V}_n} p_n^{l+2}(y, z) \leq \frac{\theta_n k_n(t)}{a_n^2} [\mathbb{E}\nu_n(u, \infty)]^2,$$

yielding $\mathbb{P}\left(\sum_{l=1}^{\theta_n-1} (III)_{2,l} \geq \eta\right) \leq \frac{\theta_n k_n(t)}{\eta a_n^2} [\mathbb{E}\nu_n(u, \infty)]^2$. The lemma now easily follows. \square

Collecting the bounds (4.19), (4.24), and (4.25), and combining them with (4.17), we obtain that under the assumptions and with the notations of Lemma 4.2, on $\Omega_{n,0}^\tau$, for all $t > 0$ and all $u > 0$,

$$\begin{aligned} & P_{\pi_n} \left(\left| \nu_n^{J,t}(u, \infty) - E_{\pi_n} [\nu_n^{J,t}(u, \infty)] \right| \geq \epsilon \right) \\ & \leq \epsilon^{-2} \left\{ \left(\frac{k_n(t)}{a_n} \right)^2 \frac{\nu_n^2(u, \infty)}{2^n} + \frac{k_n(t)}{a_n} \sigma_n^2(u, \infty) + c \frac{\nu_n(2u, \infty)}{n^2} + \rho_n [\mathbb{E}_{\pi_n} \nu_n(u, \infty)]^2 \right\}. \end{aligned} \quad (4.26)$$

for some constant $0 < c < \infty$. The proof of Proposition 4.1 is done. \square

5. Intermediate and short scales.

Set

$$\gamma_n(x) = r_n^{-1} \tau_n(x), \quad x \in \mathcal{V}_n. \quad (5.1)$$

We call $(\gamma_n(x), x \in \mathcal{V}_n)$ the re-scaled landscape. With this notation, and in the present setting, the quantities ν_n and σ_n^2 of (4.6) reads

$$\nu_n(u, \infty) = \frac{a_n}{2^n} \sum_{x \in \mathcal{V}_n} e^{-u/\gamma_n(x)}, \quad (5.2)$$

$$\sigma_n^2(u, \infty) = \frac{a_n}{2^n} \sum_{x \in \mathcal{V}_n} \sum_{x' \in \mathcal{V}_n} p_n^2(x, x') e^{-u/\gamma_n(x)} e^{-u/\gamma_n(x')}, \quad (5.3)$$

where $p_n^2(\cdot, \cdot)$ are the two steps transition probabilities of J_n .

5.1 Chain independent estimates.

In the following two propositions (Proposition 5.1 and Proposition 5.2) we collect the chain independent results needed to establish the validity of Conditions (A1), (A2), and (A3). Note that Condition (A0') reads

$$\nu_n(u, \infty)/a_n = o(1). \quad (5.4)$$

Now this will hold true as a by-product of our convergence results for $\nu_n(u, \infty)$.

Proposition 5.1: [Intermediate space scales] *Let r_n be an intermediate space scale and choose $a_n \sim b_n$ in (5.2) and (5.3). Let ν^{int} be defined in (1.25) and assume that $\beta \geq \beta_c(\varepsilon)$.*

i) If $\sum_n a_n/2^n < \infty$ then there exists a subset $\Omega_1^\tau \subset \Omega^\tau$ with $\mathbb{P}(\Omega_1^\tau) = 1$ such that, on Ω_1^τ , the following holds: for all $u > 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu_n(u, \infty) &= \nu^{int}(u, \infty), \\ \lim_{n \rightarrow \infty} n\sigma_n^2(u, \infty) &= \nu^{int}(2u, \infty), \end{aligned} \quad (5.5)$$

and $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^\delta \nu_n(u, \infty) du = 0$.

ii) If $\sum_n a_n/2^n = \infty$ then there exists a sequence of subsets $\Omega_{1,n}^\tau \subset \Omega^\tau$ with $\mathbb{P}(\Omega_{1,n}^\tau) \geq 1 - o(1)$ and such that for all n large enough, on $\Omega_{1,n}^\tau$, the following holds: for all $u > 0$

$$\begin{aligned} |\nu_n(u, \infty) - \mathbb{E}[\nu_n(u, \infty)]| &\leq 2(a_n/2^n)^{1/4} (1 \vee \sqrt{2\nu^{int}(2u, \infty)}), \\ |n\sigma_n^2(u, \infty) - n\mathbb{E}[\sigma_n^2(u, \infty)]| &\leq (a_n/2^n)^{1/4} (1 \vee \sqrt{2\nu^{int}(u, \infty)}), \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[\nu_n(u, \infty)] &= \nu^{int}(u, \infty), \\ \lim_{n \rightarrow \infty} n\mathbb{E}[\sigma_n^2(u, \infty)] &= \nu^{int}(2u, \infty), \end{aligned} \quad (5.7)$$

and, for all $0 < \delta \leq \delta_0$ and small enough δ_0 , $\int_0^\delta \nu_n(u, \infty) du \leq c_0 \delta^{1-\alpha(\varepsilon)} \frac{\alpha(\varepsilon)}{1-\alpha(\varepsilon)} \Gamma(\alpha(\varepsilon))$, for some numerical constant $0 < c_0 < \infty$.

Proposition 5.2: [Short space scales] Let r_n be an intermediate space scale and choose $a_n \sim b_n$ in (5.2) and (5.3). There exists a subset $\Omega_1^\tau \subset \Omega^\tau$ with $\mathbb{P}(\Omega_1^\tau) = 1$ such that, on Ω_1^τ , the following holds: $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^\delta \nu_n(u, \infty) du = 0$ and:

i) if $n/a_n = o(1)$ then for all $u > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu_n(u, \infty) &= 1, \\ \lim_{n \rightarrow \infty} n\sigma_n^2(u, \infty) &= 1; \end{aligned} \quad (5.8)$$

ii) if $a_n/n = \mathcal{O}(1)$ then for all $u > 0$, setting $C = 1 + \lim_{n \rightarrow \infty} a_n/n$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu_n(u, \infty) &= 1, \\ \lim_{n \rightarrow \infty} a_n \sigma_n^2(u, \infty) &= C. \end{aligned} \quad (5.9)$$

The proofs of these propositions, which are given at the end of this subsection, rely on the following three key lemmata.

Lemma 5.3: Let r_n be an intermediate or short space scale. Then for all $0 \leq \varepsilon \leq 1$ and all $0 < \beta < \infty$ satisfying $\beta \geq \beta_c(\varepsilon)$, choosing $a_n \sim b_n$ in (5.2),

$$\lim_{n \rightarrow \infty} \mathbb{E}[\nu_n(u, \infty)] = \begin{cases} \nu^{int}(u, \infty) & \text{if } 0 < \alpha(\varepsilon) \leq 1, \\ 1, & \text{if } \alpha(\varepsilon) = 0. \end{cases}, \quad u > 0. \quad (5.10)$$

where ν^{int} is defined in (1.25). Moreover, for all $L \geq 0$ such that $a_n L/2^n = o(1)$,

$$\mathbb{P} \left(|\nu_n(u, \infty) - \mathbb{E}[\nu_n(u, \infty)]| \geq 2\sqrt{a_n L/2^n} \sqrt{\mathbb{E}[\nu_n(2u, \infty)]} \right) \leq e^{-L}. \quad (5.11)$$

Lemma 5.4: $\mathbb{E}[\sigma_n^2(u, \infty)] = \frac{\mathbb{E}[\nu_n(2u, \infty)]}{n} + \frac{(\mathbb{E}[\nu_n(u, \infty)])^2}{a_n} \frac{n-1}{n}.$

Lemma 5.5: Let r_n be an intermediate or short space scale. Then for all $0 \leq \varepsilon \leq 1$ and all $0 < \beta < \infty$ satisfying $\beta \geq \beta_c(\varepsilon)$, choosing $a_n \sim b_n$ in (5.3), the following holds for all $L > 0$:

(i) if $n/a_n = o(1)$,

$$\mathbb{P} \left(|\sigma_n^2(u, \infty) - \mathbb{E}[\sigma_n^2(u, \infty)]| \geq n^{-1} \sqrt{a_n L/2^n} \sqrt{\mathbb{E}[\nu_n(u, \infty)]} \right) \leq L^{-1}, \quad (5.12)$$

(ii) if $a_n/n = \mathcal{O}(1)$,

$$\mathbb{P} \left(|\sigma_n^2(u, \infty) - \mathbb{E}[\sigma_n^2(u, \infty)]| \geq \frac{n}{a_n} \sqrt{a_n L/2^n} \sqrt{C \mathbb{E}[\nu_n(u, \infty)]} \right) \leq L^{-1}, \quad (5.13)$$

for some constant $0 < \overline{C} < \infty$.

Proof of Lemma 5.3: We first prove (5.10). By (5.2), $\mathbb{E}[\nu_n(u, \infty)] = a_n \mathbb{E}e^{-u/\gamma_n(0)}$, $0 \in \mathcal{V}_n$. For fixed $u > 0$ set $f(y) = e^{-u/y}$. Thus $f(0) = 0$, $f'(y) = (u/y^2)e^{-u/y}$ and, integrating by part,

$$\mathbb{E}[\nu_n(u, \infty)] = a_n \int_0^\infty f'(y) \mathbb{P}(\gamma_n(0) \geq y) dy = \frac{a_n}{b_n} \int_0^\infty f'(y) h_n(y) dy. \quad (5.14)$$

Set $I_n(a, b) = \int_a^b f'(y) h_n(y) dy$, $a \leq b$. Given $0 < \hat{\zeta} < 1$ and $\zeta > 1$, we may rewrite (5.14) as

$$\mathbb{E}[\nu_n(u, \infty)] = (1 + o(1)) \left[I_n(0, r_n^{-1/2}) + I_n(r_n^{-1/2}, \hat{\zeta}) + I_n(\hat{\zeta}, \zeta) + I_n(\zeta, \infty) \right], \quad (5.15)$$

where we used the assumption that $a_n/b_n \sim 1$. We will now show that, as $n \rightarrow \infty$, for small enough $\hat{\zeta}$ and large enough ζ , the leading contribution to (5.15) comes from $I_n(\hat{\zeta}, \zeta)$. To do so we first use that by (1.10) and the rough upper bound $h_n(y) \leq b_n$, $I_n(0, r_n^{-1/2}) \leq b_n \int_0^{1/\sqrt{r_n}} f'(y) dy = e^{-u\sqrt{r_n}}/\mathbb{P}(\tau_n(x) \geq r_n)$, and, together with the gaussian tail estimates (2.11), this entails

$$\lim_{n \rightarrow \infty} I_n(0, r_n^{-1/2}) = 0. \quad (5.16)$$

Next, by Lemma 2.1, (ii), with $\delta = 1/2$, $I_n(r_n^{-1/2}, \hat{\zeta}) \leq 2(1 + o(1)) \int_0^{\hat{\zeta}} f'(y) y^{-(3/4)\alpha_n} dy$ for all $0 < \hat{\zeta} < 1$, where $0 \leq \alpha_n = \alpha(\varepsilon) + o(1)$. Now, there exists $\zeta^* \equiv \zeta^*(u) > 0$ such that, for all $\hat{\zeta} < \zeta^*$, $f'(y) y^{-(3/4)\alpha_n}$ is strictly increasing on $[0, \hat{\zeta}]$. Hence, for all $\hat{\zeta} < \min(1, \zeta^*)$, $I_n(r_n^{-1/2}, \hat{\zeta}) \leq 2(1 + o(1)) u \hat{\zeta}^{-1+(3/4)[\alpha(\varepsilon)+o(1)]} e^{-u/\hat{\zeta}}$, implying that

$$\lim_{n \rightarrow \infty} I_n(r_n^{-1/2}, \hat{\zeta}) \leq 2u \hat{\zeta}^{-1+(3/4)\alpha(\varepsilon)} e^{-u/\hat{\zeta}}, \quad \hat{\zeta} < \min(1, \zeta^*). \quad (5.17)$$

To deal with $I_n(\hat{\zeta}, \zeta)$ note that by Lemma 2.1, (i), $h_n(y) \rightarrow y^{-\alpha(\varepsilon)}$, $n \rightarrow \infty$, where the convergence is uniform in $\hat{\zeta} \leq y \leq \zeta$ since, for each n , $h_n(y)$ is a monotone function, and since the limit, $y^{-\alpha(\varepsilon)}$, is continuous. Hence,

$$\lim_{n \rightarrow \infty} I_n(\hat{\zeta}, \zeta) = \lim_{n \rightarrow \infty} \int_{\hat{\zeta}}^{\zeta} f'(y) h_n(y) dy = \int_{\hat{\zeta}}^{\zeta} f'(y) y^{-\alpha(\varepsilon)} dy. \quad (5.18)$$

It remains to bound $I_n(\zeta, \infty)$. By (2.2) of Lemma 2.1, $I_n(\zeta, \infty) = \int_{\zeta}^{\infty} f'(y) h_n(y) dy = (1 + o(1)) \int_{\zeta}^{\infty} f'(y) y^{-\alpha_n} dy$, where again $0 \leq \alpha_n = \alpha(\varepsilon) + o(1)$. Thus, for $0 < \delta < 1$ arbitrary we have, taking n large enough, that for all $y \geq \zeta > 1$, $f'(y) y^{-\alpha_n} \leq f'(y) y^{-\alpha(\varepsilon) + \delta} \leq u/y^{2-\delta}$. Therefore $I_n(\zeta, \infty) \leq (1 + o(1)) \frac{1}{1-\delta} \zeta^{-(1-\delta)}$, and, choosing e.g. $\delta = 1/2$,

$$\lim_{n \rightarrow \infty} I_n(\zeta, \infty) \leq 2u\zeta^{-1/2}. \quad (5.19)$$

Collecting (5.16), (5.17), (5.18) and (5.19), we obtain that for all $\zeta > 1$ and $\hat{\zeta} < \min(1, \zeta^*)$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\nu_n(u, \infty)] = \int_{\hat{\zeta}}^{\zeta} f'(y) y^{-\alpha(\varepsilon)} dy + \mathcal{R}(\hat{\zeta}, \zeta), \quad (5.20)$$

where $0 \leq \mathcal{R}(\hat{\zeta}, \zeta) \leq +2u\hat{\zeta}^{-1+(3/4)\alpha(\varepsilon)}e^{-u/\hat{\zeta}} + 2u\zeta^{-1/2}$. Finally, passing to the limit $\hat{\zeta} \rightarrow 0$ and $\zeta \rightarrow \infty$ in (5.20) yields

$$\lim_{n \rightarrow \infty} \mathbb{E}[\nu_n(u, \infty)] = \int_0^{\infty} f'(y) y^{-\alpha(\varepsilon)} dy, \quad (5.21)$$

which is (in particular) valid for all $0 \leq \alpha(\varepsilon) \leq 1$. For $\alpha(\varepsilon) = 0$, $\int_0^{\infty} f'(y) y^{-\alpha(\varepsilon)} dy = \int_0^{\infty} f'(y) dy = 1$ while for $0 < \alpha(\varepsilon) \leq 1$, $\int_0^{\infty} f'(y) y^{-\alpha(\varepsilon)} dy = u^{-\alpha(\varepsilon)} \alpha(\varepsilon) \Gamma(\alpha(\varepsilon))$. Thus (5.10) is proven.

It remains to prove (5.11). For this we will use Bennett's bound [Ben] for the tail behavior of sums of random variables, which states that if $(X(x), x \in \mathcal{V}_n)$ is a family of i.i.d. centered random variables that satisfies $\max_x |X(x)| \leq \bar{a}$ then, setting $\bar{b}^2 = \sum_{x \in \mathcal{V}_n} \mathbb{E}X^2(x)$, for all $\bar{b}^2 \geq \tilde{b}^2$,

$$\mathbb{P}\left(\left|\sum_{x \in \mathcal{V}_n} X(x)\right| > t\right) \leq \exp\left\{\frac{t}{\bar{a}} - \left(\frac{t}{\bar{a}} + \frac{\bar{b}^2}{\bar{a}^2}\right) \log\left(1 + \frac{\bar{a}t}{\bar{b}^2}\right)\right\}, \quad t \geq 0. \quad (5.22)$$

The behavior of the r.h.s. of (5.22) varies depending on the relative size of t and of the ratio \bar{b}^2/\bar{a} . Note in particular that for $t < \bar{b}^2/(2\bar{a})$, (5.22) simplifies to

$$\mathbb{P}\left(\left|\sum_{x \in \mathcal{V}_n} X(x)\right| \geq t\right) \leq \exp\{-t^2/4\bar{b}^2\}. \quad (5.23)$$

To make use of Bennett's bound we set $X(x) = e^{-u/\gamma_n(x)} - \mathbb{E}e^{-u/\gamma_n(x)}$ so that

$$\mathbb{P}(|\nu_n(u, \infty) - \mathbb{E}[\nu_n(u, \infty)]| \geq \theta) = \mathbb{P}\left(\left|\sum_{x \in \mathcal{V}_n} X(x)\right| \geq 2^n a_n^{-1} \theta\right). \quad (5.24)$$

Since $|X(x)| \leq 1$ and $\sum_{x \in \mathcal{V}_n} \mathbb{E} X^2(x) \leq 2^n \mathbb{E} e^{-2u/\gamma_n(x)} = 2^n a_n^{-1} \mathbb{E}[\nu_n(2u, \infty)]$, we may choose $\bar{a} = 1$ and $\bar{b}^2 = 2^n a_n^{-1} \mathbb{E}[\nu_n(2u, \infty)]$. Then, by (5.23) and (5.24), for all $L > 0$,

$$\mathbb{P} \left(|\nu_n(u, \infty) - \mathbb{E}[\nu_n(u, \infty)]| \geq 2\sqrt{a_n L/2^n} \sqrt{\mathbb{E}[\nu_n(2u, \infty)]} \right) \leq e^{-L}, \quad (5.25)$$

where we chose $\theta^2 = a_n L 2^{-n+2} \mathbb{E}[\nu_n(2u, \infty)]$. This choice is permissible provided that $\theta \leq \mathbb{E}[\nu_n(2u, \infty)]/2$. In view of (5.10) this will be verified for all n large enough whenever $\theta \downarrow 0$ as $n \uparrow \infty$, i.e. whenever $a_n L/2^n = o(1)$. Thus (5.11) is established, and the lemma proven. \square

We skip the elementary proof of Lemma 5.4.

Proof of Lemma 5.5: For $u > 0$ and $l \geq 1$ set

$$\sigma_n^l(u, \infty) = a_n \sum_{y \in \mathcal{V}_n} \pi_n(y) (h_n^u(y))^l. \quad (5.26)$$

If for $l = 1$, $\sigma_n^l(u, \infty) = \nu_n(u, \infty)$ is a sum of independent random variables, this is no longer true when $l = 2$. In this case we simply resort to a second order Tchebychev inequality to write

$$\mathbb{P} (|\sigma_n^l(u, \infty) - \mathbb{E}[\sigma_n^l(u, \infty)]| \geq t) \leq t^{-2} \mathbb{E} [\sigma_n^l(u, \infty) - \mathbb{E}[\sigma_n^l(u, \infty)]]^2 = t^{-2} [\theta_1 + \theta_2], \quad (5.27)$$

where

$$\begin{aligned} \theta_1 &= \left(\frac{a_n}{2^n}\right)^2 \sum_{y \in \mathcal{V}_n} \mathbb{E} [(h_n^u(y))^l - \mathbb{E}(h_n^u(y))^l]^2, \\ \theta_2 &= \left(\frac{a_n}{2^n}\right)^2 \sum_{\substack{y, y' \in \mathcal{V}_n \times \mathcal{V}_n \\ y \neq y'}} \mathbb{E} \{ [(h_n^u(y))^l - \mathbb{E}(h_n^u(y))^l] [(h_n^u(y'))^l - \mathbb{E}(h_n^u(y'))^l] \}. \end{aligned} \quad (5.28)$$

On the one hand, we clearly have,

$$\theta_1 = \frac{a_n}{2^n} \mathbb{E}[\sigma_n^{2l}(u, \infty)] - \frac{1}{2^n} (\mathbb{E}[\sigma_n^l(u, \infty)])^2 \leq \frac{a_n}{2^n} \mathbb{E}[\sigma_n^{2l}(u, \infty)]. \quad (5.29)$$

On the other hand, after some lengthy but simple calculations, we obtain that

$$\begin{aligned} \theta_2 &\leq \frac{n(n-1)}{2^{n+1}} \left[\frac{a_n}{n^{2l}} \mathbb{E}[\nu_n(u, \infty)] + 2 \frac{(\mathbb{E}[\nu_n(u, \infty)])^2}{n^l} \left(\frac{\mathbb{E}[\nu_n(u, \infty)]}{a_n} + \frac{2}{n} \right)^{l-1} \right. \\ &\quad \left. + \frac{1}{a_n} (\mathbb{E}[\nu_n(u, \infty)])^3 \left(\frac{\mathbb{E}[\nu_n(u, \infty)]}{a_n} + \frac{1}{n} \right)^{2(l-1)} \right]. \end{aligned} \quad (5.30)$$

We now specialize the bounds (5.29) and (5.30) on θ_1 and θ_2 according to whether $n/a_n = o(1)$ or $a_n/n = \mathcal{O}(1)$. The resulting bounds will be valid under the assumptions of Lemma 5.3, and for large enough n . Assume first that $n/a_n = o(1)$. Then

$$\begin{aligned} \theta_2 &\leq \frac{\mathbb{E}[\nu_n(u, \infty)]}{2n^{2(l-1)}} \frac{a_n}{2^n} \left[1 + 2(\mathbb{E}[\nu_n(u, \infty)])^2 \frac{n}{a_n} \left(2 + \frac{n}{a_n} \mathbb{E}[\nu_n(u, \infty)] \right)^{l-1} \right. \\ &\quad \left. + \frac{1}{a_n} (\mathbb{E}[\nu_n(u, \infty)])^3 \left(\frac{n}{a_n} \right)^2 \left(2 + \frac{n}{a_n} \mathbb{E}[\nu_n(u, \infty)] \right)^{2(l-1)} \right]. \end{aligned} \quad (5.31)$$

Thus, in view of (5.10), for all n large enough,

$$\theta_2 \leq \frac{(1 + o(1))}{2} \frac{\mathbb{E}[\nu_n(u, \infty)]}{n^{2(l-1)}} \frac{a_n}{2^n}. \quad (5.32)$$

We prove in a similar way that, for all n large enough,

$$\theta_1 \leq \frac{(1 + o(1))}{n} \frac{\mathbb{E}[\nu_n(u, \infty)]}{n^{2(l-1)}} \frac{a_n}{2^n}. \quad (5.33)$$

From the last two bounds it follows that, for all n large enough,

$$\theta_1 + \theta_2 \leq \frac{\mathbb{E}[\nu_n(u, \infty)]}{n^{2(l-1)}} \frac{a_n}{2^n}. \quad (5.34)$$

Assume now that $a_n/n = \mathcal{O}(1)$ and $a_n \geq 1$. Reasoning as above it easily follows from (5.30) and (5.29) respectively that there exist constants $0 < \overline{C}, \overline{C}' < \infty$ such that, for all n large enough,

$$\theta_2 \leq \overline{C} \mathbb{E}[\nu_n(u, \infty)] \frac{a_n}{2^{n+1}} \left(\frac{n}{a_n} \right)^2, \quad (5.35)$$

and

$$\theta_1 \leq \overline{C}' \frac{a_n}{2^n} \mathbb{E}[\nu_n(u, \infty)], \quad (5.36)$$

so that

$$\theta_1 + \theta_2 \leq \overline{C} \mathbb{E}[\nu_n(u, \infty)] \frac{a_n}{2^n} \left(\frac{n}{a_n} \right)^2. \quad (5.37)$$

Inserting (5.34) in (5.27) and choosing $t = n^{-(l-1)} \sqrt{\frac{a_n L}{2^n} \mathbb{E}[\nu_n(u, \infty)]}$ yields

$$\mathbb{P} \left(\left| \sigma_n^l(u, \infty) - \mathbb{E}[\sigma_n^l(u, \infty)] \right| \geq n^{-(l-1)} \sqrt{a_n L / 2^n} \sqrt{\mathbb{E}[\nu_n(u, \infty)]} \right) \leq L^{-1}. \quad (5.38)$$

Similarly it follows from (5.37) and the choice $t = \frac{n}{a_n} \sqrt{\frac{a_n L}{2^n} \overline{C} \mathbb{E}[\nu_n(u, \infty)]}$ that

$$\mathbb{P} \left(\left| \sigma_n^l(u, \infty) - \mathbb{E}[\sigma_n^l(u, \infty)] \right| \geq \frac{n}{a_n} \sqrt{a_n L / 2^n} \sqrt{\overline{C} \mathbb{E}[\nu_n(u, \infty)]} \right) \leq L^{-1}. \quad (5.39)$$

Taking $l = 2$ in (5.38) and (5.39) give (5.12) and (5.13). (Note that the bound (5.39) is independent of l .) The proof of Lemma 5.5 is complete. \square

Proof of Proposition 5.1: By definition of an intermediate space scale, any sequence a_n such that $a_n \sim b_n$ must satisfy $a_n/2^n = o(1)$. Let us first assume that $\sum_n a_n/2^n < \infty$. This implies in particular that $(a_n \log n)/2^n = o(1)$ and $n/a_n = o(1)$. Thus, using Lemma 5.3 with $L = 2 \log n$, it follows from Borel-Cantelli Lemma that

$$\lim_{n \rightarrow \infty} \nu_n(u, \infty) = \nu^{int}(u, \infty), \quad \mathbb{P}\text{-almost surely.} \quad (5.40)$$

Together with the monotonicity of ν_n and the continuity of the limiting function ν^{int} , (5.40) entails the existence of a subset $\Omega_{1,1}^\tau \subset \Omega^\tau$ with the property that $\mathbb{P}(\Omega_{1,1}^\tau) = 1$, and such that, on $\Omega_{1,1}^\tau$,

$$\lim_{n \rightarrow \infty} \nu_n(u, \infty) = \nu^{int}(u, \infty), \quad \forall u > 0. \quad (5.41)$$

Similarly, using (5.12) of Lemma 5.5 with $L = 2^n/a_n$, it follows from Lemma 5.4 and Borel-Cantelli Lemma that

$$\lim_{n \rightarrow \infty} n\sigma_n^2(u, \infty) = \nu^{int}(2u, \infty), \quad \mathbb{P}\text{-almost surely.} \quad (5.42)$$

This and the monotonicity of σ_n^2 allows us to conclude that there exist a subset $\Omega_{1,2}^\tau \subset \Omega^\tau$ of full measure such that, on $\Omega_{1,2}^\tau$,

$$\lim_{n \rightarrow \infty} n\sigma_n^2(u, \infty) = \nu^{int}(2u, \infty), \quad \forall u > 0. \quad (5.43)$$

Finally, since the convergence is uniform in (5.41) then for each $0 < \delta \leq 1$, on $\Omega_{1,1}^\tau$, $\lim_{n \rightarrow \infty} \int_0^\delta \nu_n(u, \infty) du = \int_0^\delta \nu^{int}(u, \infty) du = \delta^{1-\alpha(\varepsilon)} \frac{\alpha(\varepsilon)}{1-\alpha(\varepsilon)} \Gamma(\alpha(\varepsilon))$. Now $\int_0^\delta \nu_n(u, \infty) du$ is a monotone increasing sequence having a continuous limit, and so, there exists a subset $\Omega_{1,3}^\tau \subset \Omega^\tau$ with the property that $\mathbb{P}(\Omega_{1,3}^\tau) = 1$, such that, on $\Omega_{1,3}^\tau$,

$$\lim_{n \rightarrow \infty} \int_0^\delta \nu_n(u, \infty) du = \int_0^\delta \nu^{int}(u, \infty) du, \quad \forall 0 < \delta \leq 1. \quad (5.44)$$

But this implies that, on $\Omega_{1,3}^\tau$, $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^\delta \nu_n(u, \infty) du = 0$. Assertion i) of the proposition now follows by taking $\Omega_1^\tau = \Omega_{1,1}^\tau \cap \Omega_{1,2}^\tau \cap \Omega_{1,3}^\tau$.

To prove Assertion ii) first note that by (5.10), given $\epsilon < 1$, there exists n_0 such that for all $n > n_0$, for all $u > 0$, $\nu_n(u, \infty) \leq \epsilon + \nu^{int}(u, \infty) \leq 1 \vee 2\nu^{int}(u, \infty)$. Using this bound and choosing $L = \sqrt{2^n/a_n}$ in (5.11), we obtain that for each fixed $u > 0$,

$$\mathbb{P}\left(|\nu_n(u, \infty) - \mathbb{E}[\nu_n(u, \infty)]| \geq 2(a_n/2^n)^{1/4}(1 \vee \sqrt{2\nu^{int}(2u, \infty)})\right) \leq \exp\{-\sqrt{2^n/a_n}\} \quad (5.45)$$

We now want to make use of Lemma 9.9 of [G1] with $X_n(u) = \nu_n(u, \infty)$, $f_n(u) = \mathbb{E}[\nu_n(u, \infty)]$, $g_n(u) \equiv g(u) = (1 \vee \sqrt{2\nu^{int}(2u, \infty)})$, $\eta_n = 2(a_n L/2^n)^{1/4}$, and $\rho_n = \exp\{-\sqrt{2^n/a_n}\}$. Indeed $(1 \vee \sqrt{2\nu^{int}(2u, \infty)})$ is a positive decreasing function, so is $\nu_n(u, \infty)$ for each n , and the properties (9.17) of Lemma 9.9 of [G1] are readily checked. Thus,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{u > 0} \left\{|\nu_n(u, \infty) - \nu^{int}(u, \infty)| \geq (a_n/2^n)^{1/4}(1 \vee \sqrt{2\nu^{int}(2u, \infty)})\right\}\right) = 0. \quad (5.46)$$

Choosing $L = \sqrt{2^n/a_n}$ in (5.12) of Lemma 5.5 we likewise obtain that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{u > 0} \left\{|n\sigma_n^2(u, \infty) - \nu^{int}(2u, \infty)| \geq (a_n/2^n)^{1/4}(1 \vee \sqrt{2\nu^{int}(2u, \infty)})\right\}\right) = 0. \quad (5.47)$$

To see that the last condition of Assertion ii) is satisfied we again make use of Lemma 9.9 of [G1], choosing this time $X_n(\delta) = \int_0^\delta \nu_n(u, \infty) du$, $f_n(\delta) = \int_0^\delta \nu^{int}(u, \infty) du$, $g_n(\delta) \equiv g(\delta) = \int_0^\delta \sqrt{2\nu^{int}(2u, \infty)} du$, and $\eta_n = 2(a_n L/2^n)^{1/4}$. Clearly, $f_n(\delta)$ and $g_n(\delta)$ are positive increasing functions for each n , and the leftmost relation in (9.17) of [G1] is satisfied, with reversed inequality, for all $l \geq 1/\delta_0$ and small enough δ_0 . Moreover, it follows from (5.46) that there exists a sequence $0 < \rho_n \downarrow 0$ such that, setting

$$A_n(\delta) = \left\{ \left| \int_0^\delta \nu_n(u, \infty) du - \int_0^\delta \nu^{int}(u, \infty) du \right| \geq 2(a_n/2^n)^{1/4} \int_0^\delta \sqrt{2\nu^{int}(2u, \infty)} du \right\}, \quad (5.48)$$

for all n large enough, for all $0 < \delta \leq \delta_0$ and small enough δ_0 , $\mathbb{P}(A_n(\delta)) \leq \rho_n$. Hence Lemma 9.9 of [G1] applies, yielding $\lim_{n \rightarrow \infty} \mathbb{P}(\sup_{0 < \delta < \delta_0} A_n(\delta)) = 0$. Now by (5.10), $\int_0^\delta \nu^{int}(u, \infty) du = \delta^{1-\alpha(\varepsilon)} \frac{\alpha(\varepsilon)}{1-\alpha(\varepsilon)} \Gamma(\alpha(\varepsilon))$ and $\int_0^\delta \sqrt{\nu^{int}(2u, \infty)} du = \delta^{1-\alpha(\varepsilon)/2} \frac{\alpha(\varepsilon)}{1-\alpha(\varepsilon)/2} \Gamma(\alpha(\varepsilon))$. Hence we have established that there exists $\Omega_{2,n}^\tau \subset \Omega^\tau$ with $\mathbb{P}(\Omega_{2,n}^\tau) \geq 1 - o(1)$ such that for all n large enough, on $\Omega_{2,n}^\tau$, for all $0 < \delta \leq \delta_0$ and small enough δ_0 ,

$$\int_0^\delta \nu_n(u, \infty) du \leq c_0 \delta^{1-\alpha(\varepsilon)} \frac{\alpha(\varepsilon)}{1-\alpha(\varepsilon)} \Gamma(\alpha(\varepsilon)), \quad (5.49)$$

for some numerical constant $0 < c_0 < \infty$. This together with (5.46) and (5.47) imply Assertion ii) of the proposition. \square

Proof of Proposition 5.2: By definition of a short space scale, any sequence a_n such that $a_n \sim b_n$ satisfies $\sum_n a_n/2^n < \infty$. Based on this observation the proof of the proposition runs along the same lines as that of assertion (i) of Proposition 5.1 with the following modifications. Since on short space scales $\alpha(\varepsilon) = 0$ it follows from (5.10) of Lemma 5.3 that $\lim_{n \rightarrow \infty} \mathbb{E}[\nu_n(u, \infty)] = 1$ (hence, the limiting function $\nu^{int}(u, \infty)$ from the statement of Proposition 5.1 is replaced by the constant 1). Next, a_n can either satisfy $n/a_n = o(1)$ or $a_n/n = \mathcal{O}(1)$. If $n/a_n = o(1)$ the last line of (5.8) follows from Lemma 5.4 and (5.12) of Lemma 5.5 exactly as the last line of (5.5) follows from them. If however $a_n/n = \mathcal{O}(1)$ then Lemma 5.4 yields $\lim_{n \rightarrow \infty} a_n \mathbb{E}[\sigma_n^2(u, \infty)] = 1 + \lim_{n \rightarrow \infty} a_n/n$. The last line of (5.9) now follows from (5.13) of Lemma 5.5, choosing e.g. $L = n^2$. \square

5.2 Proofs of the results of Section 1: the case of intermediate & short scales.

In this subsection we prove the results of Section 1 that are concerned with intermediate and short scales. These are: Assertion (i) and Assertion (ii) of Theorem 1.2 of Subsection 1.2, and Proposition 1.5 and Proposition 1.4 of Subsection 1.3. All these results rely on the central Theorem 1.8 of Subsection 1.4.

We begin with results valid for intermediate scales.

Proof of Proposition 1.5 and Assertion (ii) of Theorem 1.2: Let r_n be an intermediate space scale and assume that $\beta \geq \beta_c(\varepsilon)$. Choose $\nu = \nu^{int}$ and $a_n \sim b_n$ in Conditions (A1), (A2), and (A3) (see (1.31)–(1.33)). By the ergodic theorem of Proposition 4.1 and the chain independent estimates of Proposition 5.1, Conditions (A1), (A2), (A3) and (A0') are satisfied \mathbb{P} -almost surely if $\frac{2^{m_n}}{2^n} \log n = o(1)$, and in \mathbb{P} -probability otherwise. Thus (1.35) of

Theorem 1.8 implies that, w.r.t. the same convergence mode as above, $S_n(\cdot) \Rightarrow S^{int}(\cdot)$ where S^{int} is the subordinator of Lévy measure ν^{int} . This proves Proposition 1.5. In addition, by (1.36) of Theorem 1.8,

$$\lim_{n \rightarrow \infty} \mathcal{C}_n(t, s) = \mathcal{C}_\infty^{int}(t, s) \quad \forall 0 \leq t < t + s, \quad (5.50)$$

where $\mathcal{C}_\infty^{int}(t, s) = \mathcal{P}(\{S^{int}(u), u > 0\} \cap (t, t + s) = \emptyset)$. Assume first that $\beta > \beta_c(\varepsilon)$. Then S^{int} is a stable subordinator of index $0 < \alpha(\varepsilon) < 1$. Thus, by the Dynkin-Lamperti Theorem in continuous time (see e.g. [G1], Appendix A.2, Eq. (10.7) of Theorem 10.2), for all $t \geq 0$ and all $\rho > 0$, $\mathcal{C}_\infty^{int}(t, \rho t) = \text{Asl}_{\alpha(\varepsilon)}(1/1 + \rho)$. Assume next that $\beta = \beta_c(\varepsilon)$. Then $\alpha(\varepsilon) = 1$, implying that the range of S^{int} is the entire positive line $[0, \infty)$. Thus here $\mathcal{C}_\infty^{int}(t, s) = \mathcal{P}([0, \infty) \cap (t, t + s) = \emptyset) = 0$, for all $0 \leq t < t + s$. Taking $s = \rho t$ in (5.50) then yields (1.16). This proves Assertion (ii) of Theorem 1.2. \square

Proof of Proposition 1.4 and Assertion (i) of Theorem 1.2: Let r_n be a short space scale and let $0 < \beta < \infty$ be arbitrary. Choose $\nu = \nu^{short} = \delta_\infty$ and $a_n \sim b_n$ in Conditions (A1), (A2), and (A3). We want to follow the same strategy as before and use the ergodic theorem of Proposition 4.1, but the latter is not useful unless $a_n \gg \tau_n$. This is why we need a lower bound on r_n . Proceeding as in the proof of (2.22) we have $\log r_n = \beta \beta_c(m_n/n)n(1 + o(1))$. Thus, assuming that $\frac{1}{\beta n} \log r_n \geq 4\sqrt{\log n/n}$ implies that $m_n \geq 4\frac{\log n}{\log 2}$, so that $a_n \sim b_n = 2^{m_n} > n^4$. We now easily conclude, using Proposition 4.1 and the estimates of Proposition 5.2, that Conditions (A1), (A2), (A3) and (A0') are satisfied \mathbb{P} -almost surely. Eq. (1.35) of Theorem 1.8 then yields that, \mathbb{P} -almost surely, $S_n(\cdot) \Rightarrow S^{short}(\cdot)$ where S^{short} is the subordinator of Lévy measure ν^{short} . This proves Proposition 1.4. In addition, by (1.36) of Theorem 1.8,

$$\lim_{n \rightarrow \infty} \mathcal{C}_n(t, s) = \mathcal{C}_\infty^{short}(t, s) \quad \forall 0 \leq t < t + s, \quad (5.51)$$

where $\mathcal{C}_\infty^{short}(t, s) = \mathcal{P}(\{S^{short}(u), u > 0\} \cap (t, t + s) = \emptyset)$. Thus, since the range of S^{short} reduces to the single point 0, $\mathcal{C}_\infty^{short}(t, s) = \mathcal{P}(\{0\} \cap (t, t + s) = \emptyset) = 1$ for all $0 \leq t < t + s$. This proves Assertion (i) of Theorem 1.2 \square

6. Extreme scales.

The techniques used to deal with extreme scales differ notably from those used on shorter scales. Indeed, when $a_n \sim b_n \sim 2^n$, the convergence properties of sums such as (5.2) or (5.3) can no longer be controlled using a classical law of large number. The method we will use to do this, known as “the method of common probability space”, consists in replacing the sequence of re-scaled landscapes $(\gamma_n(x), x \in \mathcal{V}_n)$, $n \geq 1$, by a new sequence with identical distribution and almost sure convergence properties.

This section closely follows Section 7 of [G1] where this approach was first implemented. In subsection 6.1, we give an explicit representation of the re-scaled landscape which is valid for all extreme scales (Lemma 6.1) and show that, in this representation, all random variables of interest have an almost sure limit (Proposition 6.3). In subsection 6.2 we consider the model obtained by substituting the representation for the original landscape. For this model we state and prove the analogue of the ‘ergodic theorem’ of Section 4 (Proposition 6.4) and

the associated chain independent estimates of Section 5 (Proposition 6.5). Thus equipped we will be ready, in subsection 6.3, to prove the results of Section 1 that are concerned with extreme scales.

6.1 A representation of the landscape.

The representation we now introduce is due to Lepage *et al.* [LWZ] and relies on an elementary property of order statistics. We will use the following notations. Set $N = 2^n$. Let $\bar{\tau}_n(\bar{x}^{(1)}) \geq \dots \geq \bar{\tau}_n(\bar{x}^{(N)})$ and $\bar{\gamma}_n(\bar{x}^{(1)}) \geq \dots \geq \bar{\gamma}_n(\bar{x}^{(N)})$ denote, respectively, the landscape and re-scaled landscape variables $\gamma_n(x) = r_n^{-1}\tau_n(x)$, $x \in \mathcal{V}_n$, arranged in decreasing order of magnitude. As in Section 2, set $G_n(v) = \mathbb{P}(\tau_n(x) > v)$, $v \geq 0$, and denote by $G_n^{-1}(u) := \inf\{v \geq 0 : G_n(v) \leq u\}$, $u \geq 0$, its inverse. Also recall that $\alpha = \beta_c/\beta$ and assume that $\beta > \beta_c$.

Let $(E_i, i \geq 1)$ be a sequence of i.i.d. mean one exponential random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For $k \geq 1$ set

$$\begin{aligned} \Gamma_k &= \sum_{i=1}^k E_i, \\ \gamma_k &= \Gamma_k^{-1/\alpha}, \end{aligned} \tag{6.1}$$

and, for $1 \leq k \leq N$, $n \geq 1$, define

$$\gamma_n(x^{(k)}) = r_n^{-1}G_n^{-1}(\Gamma_k/\Gamma_{N+1}), \tag{6.2}$$

where $\{x^{(1)}, \dots, x^{(N)}\}$ is a randomly chosen labelling of the N elements of \mathcal{V}_n , all labellings being equally likely.

Lemma 6.1: *For each $n \geq 1$, $(\bar{\gamma}_n(\bar{x}^{(1)}), \dots, \bar{\gamma}_n(\bar{x}^{(N)})) \stackrel{d}{=} (\gamma_n(x^{(1)}), \dots, \gamma_n(x^{(N)}))$.*

Proof: Note that G_n is non-increasing and right-continuous so that G_n^{-1} is non-increasing and right-continuous. It is well known that if the random variable U is a uniformly distributed on $[0, 1]$ we may write $\tau_n(0) \stackrel{d}{=} G_n^{-1}(U)$ (see e.g. [Re], page 4). In turn it is well known (see [Fe], Section III.3) that if $(U(k), 1 \leq k \leq N)$ are independent random variables uniformly distributed on $[0, 1]$ then, denoting by $\bar{U}_n(1) \leq \dots \leq \bar{U}_n(N)$ their ordered statistics, $(\bar{U}_n(1), \dots, \bar{U}_n(N)) \stackrel{d}{=} (\Gamma_1/\Gamma_{N+1}, \dots, \Gamma_N/\Gamma_{N+1})$. Combining these two facts readily yields the claim of the lemma since, by independence of the landscape variables $\tau_n(x)$, all arrangements of the N variables Γ_k/Γ_{N+1} on the N vertices of \mathcal{V}_n are equally likely. \square

Next, let Υ be the point process in $M_P(\mathbb{R}_+)$ which has counting function

$$\Upsilon([a, b]) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\gamma_k \in [a, b]\}}. \tag{6.3}$$

Lemma 6.2: *Υ is a Poisson random measure on $(0, \infty)$ with mean measure μ given by (1.18).*

Proof: The point process $\Gamma = \sum_{i=1}^{\infty} \mathbb{I}_{\{\Gamma_k\}}$ defines a homogeneous Poisson random measure on $[0, \infty)$ and thus, by the mapping theorem ([Re], Proposition 3.7), setting $T(x) = x^{-1/\alpha}$ for $x > 0$, $\Upsilon = \sum_{i=1}^{\infty} \mathbb{I}_{\{T(\Gamma_k)\}}$ is Poisson random measure on $(0, \infty)$ with mean measure $\mu(x, \infty) = T^{-1}(x)$. \square

We thus established that both the ordered landscape variables and the point process Υ can be expressed in terms of the common sequence $(E_i, i \geq 1)$ and thus, on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. This is central to the proof of the next proposition.

Proposition 6.3: *Assume that $\alpha < 1$. Let r_n be an extreme space scale. Let $f : (0, \infty) \rightarrow [0, \infty)$ be a continuous function that obeys*

$$\int_{(0, \infty)} \min(f(u), 1) d\mu(u) < \infty. \quad (6.4)$$

Then, \mathbf{P} -almost surely,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^N f(\gamma_n(x^{(k)})) = \sum_{k=1}^{\infty} f(\gamma_k) < \infty. \quad (6.5)$$

The proof of Proposition 6.3 closely follows that of Proposition 7.3 of [G1], which itself is strongly inspired from the proof of Proposition 3.1 of [FIN].

Proof of Proposition 6.3: Lemma 2.2 of Section 2 will come into use now. By the strong law of large numbers there exists a subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all n large enough and all $\omega \in \tilde{\Omega}$, $\Gamma_{N+1} = b_n(1 + \lambda_n)$ where $\lambda_n = o(1)$. From now on we assume that $\omega \in \tilde{\Omega}$. Thus

$$\sum_{i=1}^N f(\gamma_n(x^{(i)})) = \sum_{i=1}^N f(g_n(\Gamma_i/(1 + \lambda_n))) . \quad (6.6)$$

where g_n is defined as in (2.4). Let us first consider the case $f(x) = x$, $x > 0$. Recall the notation $\gamma_i = \Gamma_i^{-1/\alpha}$. For $y > 0$ set $I(y) = \{i \geq 1 : \gamma_i \geq y\}$, $I^c(y) = \{i \geq 1 : \gamma_i < y\}$ and, for $\kappa_n = (b_n(1 - \Phi(1/(\beta\sqrt{n}))))^{-1/\alpha}$, $\delta > 0$, and large enough n , write:

$$\sum_{i=1}^N \gamma_n(x^{(i)}) = \sum_{i \in I(\delta)} \gamma_n(x^{(i)}) + \sum_{i \in I(\kappa_n) \setminus I(\delta)} \gamma_n(x^{(i)}) + \sum_{i \in I^c(\kappa_n)} \gamma_n(x^{(i)}). \quad (6.7)$$

From assertion (i) of Lemma 2.2 it follows that,

$$\sum_{i \in I(\delta)} \gamma_n(x^{(i)}) \rightarrow \sum_{i \in I(\delta)} \gamma_i, \quad n \rightarrow \infty. \quad (6.8)$$

Next, by assertion (ii) of Lemma 2.2, for all $0 < \delta < 1$ and some constant $0 < C < \infty$, we have

$$\sum_{i \in I(n^{-1/\alpha}) \setminus I(\delta)} \gamma_n(x^{(i)}) \leq \sum_{i \in I(\kappa_n) \setminus I(\delta)} C\Gamma_i^{-1/\alpha} = \sum_{i \in I(\kappa_n) \setminus I(\delta)} C\gamma_i. \quad (6.9)$$

The last sum in (6.9) is bounded above by $W_\delta = \sum_{i:\gamma_i \leq \delta} C\gamma_i$, and, proceeding as in (6.18)-(6.19) of [G1] one gets that, choosing $d > 0$ such that $\delta + \alpha W := \lim_{\delta \rightarrow 0} W_\delta = 0$ \mathbf{P} -almost surely. Finally, for $i \in I^c(\kappa_n)$, $\Gamma_i/b_n \leq 1 - \Phi(1/(\beta\sqrt{n}))$. Being the inverse of the tail of a probability distribution, $G_n^{-1}(x) \downarrow 0$ as $x \uparrow 1$, and $G_n^{-1}(x) = 0$ for all $x \geq 1$. From the calculations of the proof of Lemma 2.2 we see that for small $\epsilon > 0$, $G_n^{-1}(1 - \epsilon) \approx \exp(-\beta\sqrt{2n \log(1/\epsilon)})$. Thus, for n large enough,

$$G_n^{-1}(\Gamma_i/[b_n(1 + \lambda_n)]) \leq G_n^{-1}([1 - \Phi(1/(\beta\sqrt{n}))]/[1 + \lambda_n]) \leq 1, \quad (6.10)$$

so that

$$\sum_{i \in I^c(\kappa_n)} \gamma_n(x^{(i)}) \leq \sum_{i \in I^c(\kappa_n)} r_n^{-1} \leq b_n r_n^{-1}. \quad (6.11)$$

Clearly $b_n r_n^{-1} \downarrow 0$ as $n \uparrow \infty$ since $b_n = 2^{m_n} = 2^{n(1-o(1))}$, whereas by (2.24), $r_n = e^{(\beta\beta_c n[1-o(1)])}$ where by assumption on β , $\beta\beta_c \geq \beta_c^2 = 2 \log 2$. Together with (6.11) this yields,

$$\sum_{i \in I^c(\kappa_n)} \gamma_n(x^{(i)}) \rightarrow 0, \quad n \rightarrow \infty. \quad (6.12)$$

Combining the previous estimates we obtain that, on a subset of Ω of full measure,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^N \gamma_n(x^{(i)}) = \lim_{\delta \rightarrow 0} \sum_{i:\gamma_i \geq \delta} \gamma_i = \sum_{i=1}^{\infty} \gamma_i. \quad (6.13)$$

The proof of (6.5) goes along the same line. We refer the reader to the proof of (6.6) of [G1] for details. This concludes the proof of Proposition 6.3. \square

6.2 Preparations to the verification of Conditions (A1), (A2), and (A3).

Consider the model obtained by substituting the representation $(\gamma_n(x^{(i)}), 1 \leq i \leq N)$ for the original re-scaled landscape $(\gamma_n(x), x \in \mathcal{V}_n)$. The aim of this subsection is to prove the homologue, for this model, of the ‘ergodic theorem’ (Proposition 4.1) and chain independent estimates (Proposition 5.1) of Section 4 and Section 5.

In order to distinguish the quantities $\nu_n^{J,t}(u, \infty)$, $(\sigma_n^{J,t})^2(u, \infty)$, $\nu_n(u, \infty)$ and $\sigma_n^2(u, \infty)$, expressed in (4.1)–(4.6) in the original landscape variables, from their expressions in the new ones, we call the latter $\mathbf{v}_n^{J,t}(u, \infty)$, $(\mathbf{s}_n^{J,t})^2(u, \infty)$, $\mathbf{v}_n(u, \infty)$ and $\mathbf{s}_n^2(u, \infty)$ respectively. Their definition is otherwise unchanged.

Proposition 6.4: *There exists a subset $\Omega_0 \subset \Omega$ such that $\mathbf{P}(\Omega_0) = 1$ and such that, on Ω_0 , for all large enough n , the following holds: for all $t > 0$ and all $u > 0$,*

$$P_{\pi_n}(|\mathbf{v}_n^{J,t}(u, \infty) - E_{\pi_n}[\mathbf{v}_n^{J,t}(u, \infty)]| \geq \epsilon) \leq \epsilon^{-2} \Theta_n(t, u), \quad \forall \epsilon > 0, \quad (6.14)$$

where

$$\begin{aligned} \Theta_n(t, u) = & \frac{k_n(t)}{a_n} \left(\frac{\mathbf{v}_n^2(u, \infty)}{2^n} + \mathbf{s}_n^2(u, \infty) \right) + c_1 \frac{\mathbf{v}_n(2u, \infty)}{n^2} \\ & + \frac{k_n(t)}{a_n} \left(3\theta_n e^{-u/\delta_n} \mathbf{v}_n(u, \infty) + \frac{2^n}{a_n} \mathbf{v}_n^2(u, \infty) e^{-c_2 u} \right), \end{aligned} \quad (6.15)$$

for some constants $0 < c_1, c_2 < \infty$, where $\delta_n \leq n^{-\alpha(1+o(1))}$, and where θ_n is defined as in Proposition 3.1. In addition, for all $t > 0$ and all $u > 0$,

$$P_{\pi_n}((\mathbf{s}_n^{J,t})^2(u, \infty) \geq \epsilon') \leq \frac{k_n(t)}{\epsilon' a_n} \mathbf{s}_n^2(u, \infty), \quad \forall \epsilon' > 0. \quad (6.16)$$

Proposition 6.5: Let r_n be an extreme space scale and choose $a_n \sim b_n$. Assume that $\beta \geq \beta_c$ let ν^{ext} be defined in (1.28). There exists a subset $\Omega_1 \subset \Omega$ such that $\mathbf{P}(\Omega_1) = 1$ and such that, on Ω_1 , the following holds: for all $u > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{v}_n(u, \infty) &= \nu^{ext}(u, \infty) < \infty, \\ \lim_{n \rightarrow \infty} \mathbf{s}_n^2(u, \infty) &= 0, \end{aligned} \quad (6.17)$$

and $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^\delta \mathbf{v}_n(u, \infty) = 0$.

Proposition 6.5 is a straightforward application of Proposition 6.3 and Lemma 1.7 whose proof we skip (see also [G1], (6.32)-(6.35) for a pattern of proof).

Proof of Proposition 6.4: Proposition 6.4 is a rerun of the proof Proposition 4.1. The only difference is in the treatment of the term (4.23). In the new landscape variables, Lemma 4.2 is not true, and its method of proof is unadapted. To bound (4.23) we proceed as follows. Let $T_n := \{x^{(k)}, 1 \leq k \leq n\} \subset \mathcal{V}_n$ be the set of the n vertices with largest $\gamma_n(x)$. The next two lemmata collect elementary properties of T_n .

Lemma 6.6: There exists a subset $\Omega_{0,1} \subset \Omega$ with $\mathbf{P}(\Omega_{0,1}) = 1$ such that, for all $\omega \in \Omega_{0,1}$, for all large enough n , the following holds: for all $x, x' \in T_n$, $x \neq x'$, $\text{dist}(x, x') = \frac{n}{2}(1 - \rho_n)$ where $\rho_n = \sqrt{\frac{8 \log n}{n}}$.

Proof: Given $t > 0$ consider the event $\Omega_{0,1}(n) = \left\{ \exists_{1 \leq k \neq k' \leq n} : \left| \text{dist}(x^{(k)}, x^{(k')}) - \frac{n}{2} \right| \geq t \right\}$. By construction, the elements of T_n are drawn at random from the \mathcal{V}_n , independently and without replacement. Hence

$$\mathbf{P}(\Omega_{0,1}(n)) \leq n^2 \mathbf{P}\left(\left| \text{dist}(x^{(1)}, x^{(2)}) - \frac{n}{2} \right| \geq t\right) \sim n^2 P\left(\left| \sum_{i=1}^n \varepsilon_i - \frac{n}{2} \right| \geq t\right), \quad (6.18)$$

where $(\varepsilon_i, 1 \leq i \leq n)$ are i.i.d. r.v.'s taking value 0 and 1 with probability 1/2. A Classical exponential Tchebychev inequality yields $P\left(\left| \sum_{i=1}^n \varepsilon_i - \frac{n}{2} \right| \geq t\right) \leq e^{-\frac{t^2}{2n}}$. Choosing $t = \sqrt{8n \log n}$, and plugging into (6.18), $\mathbf{P}(\Omega_{0,1}(n)) \leq n^{-2}$. Setting $\Omega_{0,1} = \bigcup_{n_0} \bigcap_{n > n_0} \Omega_{0,1}(n)$, the claim of the lemma follows from an application of Borel-Cantelli Lemma. \square

Lemma 6.7: There exists a subset $\Omega_{0,2} \subset \Omega$ with $\mathbf{P}(\Omega_{0,2}) = 1$ such that, for all $\omega \in \Omega_{0,2}$, for all large enough n , $\sup\{\gamma_n(x), x \in \mathcal{V}_n \setminus T_n\} \leq \delta_n$ where $\delta_n = (1 + o(1))n^{-\alpha(1+o(1))}$.

Proof: Clearly $\sup\{\gamma_n(x), x \in \mathcal{V}_n \setminus T_n\} = \sup\{\gamma_n(x^{(k)}), k > n\} = \gamma_n(x^{n+1}) = r_n^{-1} G_n^{-1}\left(\frac{\Gamma_{n+1}}{\Gamma_{N+1}}\right)$, where the last equality follows from (6.2). Reasoning as in the paragraph preceding (6.6), but applying the strong law of large numbers to both Γ_{n+1} and Γ_{N+1} , we deduce that there exists a subset $\Omega_{0,2} \subset \Omega$ of full measure such that, for all n large enough and all $\omega \in \Omega_{0,2}$,

$\gamma_n(x^{n+1}) = r_n^{-1}G_n^{-1}((n/b_n)(1+\lambda_n))$. By definition of $h_n(v)$ (see (2.1)), $r_n^{-1}G_n^{-1}(h_n(v)) = v$, and by Lemma 2.1, $\gamma_n(x^{n+1}) = (1+o(1))n^{-\alpha(1+o(1))}$. \square

We are now equipped to bound $(III)_{2,l}$. Set $\Omega_0 = \Omega_{0,1} \cap \Omega_{0,2}$. Writing $T_n^c \equiv \mathcal{V}_n \setminus T_n$, and setting $f(y, z) = k_n(t)\pi_n(y)e^{-u[\gamma_n^{-1}(y)+\gamma_n^{-1}(z)]}p_n^{l+2}(y, z)$, we may decompose $(III)_{2,l}$ it into four terms,

$$\sum_{z \in T_n^c, y \in T_n^c: y \neq z} f(y, z) + \sum_{z \in T_n^c, y \in T_n} f(y, z) + \sum_{z \in T_n, y \in T_n^c} f(y, z) + \sum_{z \in T_n, y \in T_n: y \neq z} f(y, z). \quad (6.19)$$

To bound the first sum above we use that, by Lemma 6.7, for $y \in T_n^c$, $e^{-u[\gamma_n^{-1}(z)+\gamma_n^{-1}(y)]} \leq e^{-u/\gamma_n(z)}e^{-u/\delta_n}$. Thus,

$$\begin{aligned} \sum_{z \in T_n^c, y \in T_n^c: y \neq z} f(y, z) &\leq e^{-u/\delta_n} \sum_{z \in T_n^c} k_n(t)\pi_n(z)e^{-u/\gamma_n(z)} \sum_{y \in T_n^c: y \neq z} p_n^{l+2}(y, z) \\ &\leq e^{-u/\delta_n} \sum_{z \in T_n^c} k_n(t)\pi_n(z)e^{-u/\gamma_n(z)} \\ &\leq e^{-u/\delta_n} \frac{k_n(t)}{a_n} \mathbf{v}_n(u, \infty). \end{aligned} \quad (6.20)$$

The second and third sums of (6.19) are bounded just in the same way. To deal with the last sum we use that in view of Lemma 6.6 the assumptions of Proposition 3.3 are satisfies. Consequently

$$\begin{aligned} \sum_{l=1}^{\theta_n-1} \sum_{z \in T_n, y \in T_n: y \neq z} f(y, z) &\leq \frac{2^n k_n(t)}{a_n^2} \left[a_n \sum_{z \in T_n} \pi_n(z)e^{-u/\gamma_n(z)} \right]^2 \sum_{l=1}^{\theta_n-1} p_n^{l+2}(y, z), \\ &\leq e^{-cn} \frac{2^n k_n(t)}{a_n^2} (\mathbf{v}_n(u, \infty))^2, \end{aligned} \quad (6.21)$$

for some constant $0 < c < \infty$. Collecting (6.19), (6.20) and (6.21), and summing over l , we finally get,

$$\sum_{l=1}^{\theta_n-1} (III)_{2,l} \leq 3\theta_n e^{-u/\delta_n} \frac{k_n(t)}{a_n} \mathbf{v}_n(u, \infty) + e^{-cn} \frac{2^n k_n(t)}{a_n^2} (\mathbf{v}_n(u, \infty))^2. \quad (6.22)$$

Proposition 6.4 is now proved just as Proposition 4.1, using the bound (6.22) instead of the bound (4.25) of Lemma 4.2. \square

6.3 Proofs of the results of Section 1: the case of extreme scales.

We now prove the results of Section 1 that are concerned with extreme scales, namely: Proposition 1.6, Assertion (iii) of Theorem 1.2, Theorem 1.3, and Lemma 1.7. Again our key tool will be Theorem 1.8 of Subsection 1.4.

We assume throughout this section that r_n is an extreme space scale and that $\beta \geq \beta_c$.

Proof of Proposition 1.6 and Assertion (iii) of Theorem 1.2: Consider the model obtained by substituting the representation $(\gamma_n(x^{(i)}), 1 \leq i \leq N)$ for the original landscape $(\gamma_n(x), x \in \mathcal{V}_n)$. Let $\tilde{\mathbf{S}}_n(\cdot)$, $\mathbf{S}_n(\cdot)$, and $\mathbf{C}_n(t, s)$ denote, respectively, the clock process (1.2), the re-scaled clock process (1.22), and the time correlation function (1.8) expressed in the new landscape variables. Choose $\nu = \nu^{ext}$ and $a_n \sim b_n$ in Conditions (A1), (A2), and (A3) (that is, in (1.31), (1.32), and (1.33), expressed of course in the new landscape variables). By Proposition 6.4 and Proposition 6.5, there exists a subset $\Omega_2 \subset \Omega$ with $\mathbf{P}(\Omega_2) = 1$, such that, on Ω_2 , Conditions (A1), (A2), (A3), and (A0') are satisfied. By (1.35) of Theorem 1.8 we thus have that, on Ω_2 , $\mathbf{S}_n(\cdot) \Rightarrow S^{ext}(\cdot)$ where S^{ext} is the (random) subordinator of Lévy measure ν^{ext} . This proves Proposition 1.6.

To prove Assertion (iii) of Theorem 1.2 first note that by Lemma 6.1,

$$\mathcal{C}_n(t, s) \stackrel{d}{=} \mathbf{C}_n(t, s) \quad \text{for all } n \geq 1 \text{ and all } 0 \leq t < t + s. \quad (6.23)$$

Next, by (1.36) of Theorem 1.8 we have that, on Ω_2 ,

$$\lim_{n \rightarrow \infty} \mathbf{C}_n(t, s) = \mathcal{C}_\infty^{ext}(t, s) \quad \forall 0 \leq t < t + s, \quad (6.24)$$

where $\mathcal{C}_\infty^{ext}(t, s) = \mathcal{P}(\{S^{ext}(u), u > 0\} \cap (t, t + s) = \emptyset)$. Now, by Lemma 1.7, there exists a subset $\Omega_3 \subset \Omega$ with $\mathbf{P}(\Omega_3) = 1$, such that, on Ω_3 , ν^{ext} is regularly varying at infinity with index $-\alpha$. Thus, by Dynkin-Lamperti Theorem in continuous time applied for fixed $\omega \in \Omega_3$ (see e.g. [G1], Appendix A.2, Eq. (10.6) of Theorem 10.2) we get that,

$$\lim_{t \rightarrow 0+} \mathcal{C}_\infty^{ext}(t, \rho t) = \text{Asl}_\alpha(1/1 + \rho) \quad \forall \rho > 0. \quad (6.25)$$

Thus, by (6.23) with $s = \rho t$, using successively (6.24) and (6.25) to pass to the limit $n \rightarrow \infty$ and $t \rightarrow 0+$, we obtain that, for all $\rho > 0$, $\lim_{t \rightarrow 0+} \lim_{n \rightarrow \infty} \mathcal{C}_n^{ext}(t, \rho t) \stackrel{d}{=} \text{Asl}_\alpha(1/1 + \rho)$. Since convergence in distribution to a constant implies convergence in probability, the claim of Theorem 1.2, (iii) follows. \square

Proof of Theorem 1.3: The proof of Theorem 1.3 is a re-run of the proof of Theorem 4.5 of [G1] (setting $a = 0$). Note indeed that for all $\beta > \beta_c$, $0 \leq \alpha \leq 1$, which implies that $\int_0^\infty \nu^{ext}(u, \infty) du = \sum_{k=1}^\infty \gamma_k < \infty$ \mathbf{P} -almost surely. We are thus in the realm of “classical” renewal theory, in the so-called “finite mean life time” case. The second and first assertions of Theorem 1.3 will then follow, respectively, from Theorem 10.2, (ii), and Theorem 10.4, (ii), of appendix A.2 of [G1]. Their proofs use the following two elementary observations:

$$\frac{\int_s^\infty \nu^{ext}(u, \infty) du}{\int_0^\infty \nu^{ext}(u, \infty) du} = \mathcal{C}_\infty^{sta}(s), \quad u > 0, \quad (6.26)$$

where \mathcal{C}_∞^{sta} is defined in (1.19); Moreover, setting

$$1 - F_n(v) := \sum_{x \in \mathcal{V}_n} \mathcal{G}_{\alpha, n}(x) e^{-v c_n \lambda_n(x)} = \sum_k \frac{\gamma_n(x^{(k)})}{\sum_l \gamma_n(x^{(l)})} e^{-s/\gamma_n(x^{(l)})},$$

a simple application of Proposition 6.3 yields, $\lim_{n \rightarrow \infty} (1 - F_n(v)) = (1 - F^{sta}(v)) := \mathcal{C}_\infty^{sta}(s)$ \mathbf{P} -almost surely. We refer the reader to [G1] for details. \square

It now remains to prove Lemma 1.7.

Proof of Lemma 1.7: To ease the notation set $\varepsilon' = 1$. Set $u^{-\alpha} = M$ and $f(x) = e^{-1/x}$. By (1.28) we may write

$$u^\alpha \nu^{ext}(u, \infty) = \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k). \quad (6.27)$$

The lemma will thus be proven if we can prove that:

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) = \alpha \Gamma(\alpha) \quad \mathbf{P}\text{-almost surely.} \quad (6.28)$$

Note that it is enough for this to take the limit along the integers since, $f(M^{1/\alpha} \gamma_k)$ being a strictly increasing function of M ,

$$\frac{\lfloor M \rfloor}{M} \frac{1}{\lfloor M \rfloor} \sum_{k=1}^{\infty} f(\lfloor M \rfloor^{1/\alpha} \gamma_k) \leq \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) \leq \frac{\lceil M \rceil}{M} \frac{1}{\lceil M \rceil} \sum_{k=1}^{\infty} f(\lceil M \rceil^{1/\alpha} \gamma_k). \quad (6.29)$$

The claim will now follow from a classical large deviation upper bound. Set

$$A_M = \left\{ \left| \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) - \mathbf{E} \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) \right| \geq \delta_M \right\}, \quad (6.30)$$

where δ_M is defined through $\delta_M^2 = 4\alpha\Gamma(\alpha) \frac{\log M}{M}$. By Tchebychev exponential inequality, for all $\lambda > 0$,

$$\mathbf{P}(A_M) \leq 2 \exp \left\{ -\lambda \delta_M - \mathbf{E}(\lambda/M) \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) + \log \mathbf{E} \exp \left\{ (\lambda/M) \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) \right\} \right\}. \quad (6.31)$$

Simple Poisson point process calculations yield $\mathbf{E} \frac{1}{M} \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) = \alpha \Gamma(\alpha)$ and

$$\log \mathbf{E} \exp \left\{ (\lambda/M) \sum_{k=1}^{\infty} f(M^{1/\alpha} \gamma_k) \right\} = - \int_0^\infty (1 - e^{-\frac{\lambda}{M} f(M^{1/\alpha} x)}) d\mu(x). \quad (6.32)$$

Furthermore, for all $k \geq 1$, $\int_0^\infty f^k(M^{1/\alpha} x) d\mu(x) = k^{-\alpha} M \alpha \Gamma(\alpha)$. Thus

$$- \int_0^\infty (1 - e^{-\frac{\lambda}{M} f(M^{1/\alpha} x)}) d\mu(x) \leq \alpha \Gamma(\alpha) \left(\lambda + \frac{\lambda^2}{4M} e^{\frac{\lambda}{2M}} \right). \quad (6.33)$$

From this last bound and the choice $\lambda = \delta_M \frac{2M}{\alpha \Gamma(\alpha)}$, (6.31) yields

$$\mathbf{P}(A_M) \leq 2 \exp \left\{ -\frac{\delta_M^2 M}{\alpha \Gamma(\alpha)} \left(2 - e^{2\delta_M / \alpha \Gamma(\alpha)} \right) \right\} \leq \frac{2}{M^2}, \quad (6.34)$$

where the last inequality follows from the definition of δ_M . Thus $\sum_M \mathbf{P}(A_M) \leq \infty$, and this and the first Borel-Cantelli Lemma prove (6.28). \square

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